CSCE 2211 Dr. A. Goneid

Some Useful Mathematical Relations

Functional Relations

- $n^x = 2^{x \log n}$ • $(x^y)^z = x^{yz}$ • $x^y x^z = x^{y+z}$ • $\log(xy) = \log x + \log y$
- $2^{\log n} = n$ • $\log(n^x) = x \log n$ • $\log_b n = \log_a n / \log_a b$
- Stirling's Theorem: $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ for large n
- $\log n! = c_1 + c_2 \log n + n \log n - c_3 n = O(n \log n)$
- $\sum_{i=1}^{n} \log i = \log n! = O(n \log n)$
- $\sum_{i=1}^{n} i \log i = \sum_{i=1}^{n} \log i^i \le n \log n^n = O(n^2 \log n)$

Finite Series

- **Arithmetic Series:** $\sum_{i=0}^{n-1} (a + b\,i) = (n/2)\,\{2a + (n-1)b\}$

  $\sum_{i=0}^{n-1} (n-i) = \sum_{i=1}^{n} i = n(n+1)/2$

- **Geometric Series:** $\sum_{i=0}^{n} x^i = (x^{n+1} - 1)/(x-1)$ for $x \neq 1$

  $\sum_{i=0}^{n-1} 2^i = 2^n - 1$ $\sum_{i=1}^{n-1} 2^i = 2^n - 2$ $\sum_{i=1}^{n} i\, 2^{i-1} = (n-1)\, 2^n + 1$

$\sum_{i=0}^{n-1} i\, 2^{n-i} = \sum_{i=1}^{n} (n-i) 2^i = 2^{n+1} - 2(n+1)$

Sums of Powers of Natural Numbers

- $\sum_{i=1}^{n} i = n(n+1)/2$
- $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$

- $\sum_{i=1}^{n} i^3 = [n(n+1)/2]^2$
- $\sum_{i=1}^{n} i^4 = n(n+1)(2n+1)(3n^2 + 3n - 1) / 30$
- $\sum_{i=1}^{n} i^5 = n^2 (n+1)^2 (2 n^2 + 2n - 1) / 12$

Sums of Odd Numbers and their Powers

- $\sum_{i=1}^{n} (2 i - 1) = n^2$
- $\sum_{i=1}^{n} (2 i - 1)^2 = n(4n^2 - 1)/3$
- $\sum_{i=1}^{n} (2 i - 1)^3 = n^2 (2n^2 - 1)$

Other Finite Series

- $\sum_{i=1}^{n} i (i + 1)^2 = (1/12) n(n+1)(n+2)(3n + 5)$
- $\sum_{i=1}^{n} i . i! = (n+1)! - 1$
- $\sum_{i=2}^{n} 1 /(i^2 - 1) = (3/4) - (2n+1)/[2n(n+1)]$
- $\sum_{i=1}^{n} 1/ i \sim \gamma + \ln n + 1/(2n)$ for large n, $\gamma = 0.577$ = Euler's constant
- $\sum_{i=1}^{n} \frac{1}{i (i+1)} = \frac{n}{n+1}$

Infinite Series and Products

- $\sum_{k=0}^{\infty} a^{kx} = 1 / (1 - a^x)$ for $(a > 1$ and $x < 0)$ or $(0 < a < 1$ and $x > 0)$

- $\sum_{k=0}^{\infty} x^k / k! = e^x$
- $\sum_{k=0}^{\infty} x^k = 1 / (1-x)$ for $|x| < 1$
- $\sum_{k=0}^{\infty} (a + bk)\, x^k = a/(1-x) + b\,x/(1 - x^2)$ for $|x| < 1$
- $\sum_{k=1}^{\infty} (-1)^{k+1} / k = \ln 2$
- $\sum_{k=1}^{\infty} (-1)^{k+1} / k^2 = \pi^2 / 12$
- $\sum_{k=1}^{\infty} 1 / (2k - 1)^2 = \pi^2 / 8$
- $\sum_{k=1}^{\infty} 1 / (k\, 2^k) = \ln 2$
- $\sum_{k=1}^{\infty} 1 / (k^2\, 2^k) = \pi^2 / 12 - (\ln 2)^2 / 2$
- $\sum_{k=0}^{\infty} \frac{1}{k\,!} = e \quad = 2.71828$
- $\sum_{k=0}^{\infty} \frac{(-1)^k}{k\,!} = 1/e \quad = 0.36787$
- $\sum_{k=1}^{\infty} k / (k+1)! = 1$
- $\prod_{k=2}^{\infty} \left(1 - \frac{1}{k^2}\right) = \frac{1}{2}$
- $\prod_{k=0}^{\infty} (1 + x^{2^k}) = \frac{1}{1-x}$ for $|x| < 1$