Asymmetric Degrees of Freedom of the Full-Duplex MIMO 3-Way Channel

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***Abstract*—In this paper, we characterize the asymmetric total degrees of freedom (DoF) of a multiple-input multiple-output (MIMO) 3-way channel. Each node has a separate-antenna full-duplex MIMO transceiver with a different number of antennas, where each antenna can be configured for either signal transmission or reception. Each node has two unicast messages to be delivered to the two other nodes. We first derive upper bounds on the total DoF of the system. Cut-set bounds in conjunction with genie-aided bounds are derived to characterize the achievable total DoF. Afterwards, we analytically derive the optimal number of transmit and receive antennas at each node to maximize the total DoF of the system, subject to the total number of antennas at each node. Finally, the achievable schemes are constructed. The proposed schemes are mainly based on zero-forcing and null-space transmit beamforming.**

I. Introduction

Full-duplex systems have attracted a great deal of attention recently due to their potential benefits to significantly enhance the throughput and spectral efficiency of conventional half-duplex systems. Recent results from academia [1, and references therein] and industry [2] have proposed various practical designs to implement in-band full-duplex radios by cancelling or suppressing the self-interference signal, generated during simultaneous transmission and reception, at the RF and baseband level. There are two possible methods of antenna interfacing for full-duplex MIMO transceivers; separate-antenna architecture, and shared-antenna architecture [1, Section IV]. In the separate-antenna architecture, each antenna is dedicated to either signal transmission or reception. In the shared-antenna architecture, each antenna simultaneously transmits and receives signals on the same channel with the aid of a circulator that routes the transmitted signal from the TX signal chain to the antenna and the received signal on the antenna to the RX signal chain.

The two-way communication channel was introduced in the seminal paper by Shannon [3]. The extension of the 2-way channel to the case of three nodes, i.e., the 3-way channel, has recently attracted much attention [4], [5]. It is assumed that all nodes operate in a perfect full-duplex mode. Furthermore, there are six unicast messages to be exchanged among the nodes; each node is intended to exchange unicast messages with the other nodes simultaneously. The sum-capacity of the 3-way channel, that characterizes the DoF of the channel, is studied in [4] with respect to the Gaussian channel model. The authors in [5] investigate the symmetric DoF of a MIMO 3-way channel with a homogeneous number of antennas; each node has M_T transmit antennas and M_R receive antennas.

The main contribution of this paper is the characterization of the asymmetric total DoF of a MIMO 3-way channel. Each node has a separate-antenna full-duplex MIMO transceiver where each antenna can be configured to either transmit or receive, and the nodes have a different number of antennas. Each node has two unicast messages to be delivered to the two other nodes. It should be noted that the proposed system model is a generalized version of the symmetric model studied by Maier *et al.* in [5] where the total number of antennas of each node are the same, and each node has M_T transmit antennas and M_R receive antennas.

We first derive upper bounds on the total DoF of the system in terms of M_{T_ℓ} and M_{R_ℓ}, where $\ell \in \{1, 2, 3\}$. Cut-set bounds in conjunction with genie aided bounds are utilized to characterize the achievable total DoF. Afterwards, we analytically derive the optimal number of transmit and receive antennas at each node to maximize the total DoF of the system, subject to the total number of antennas at each node. Finally, the achievable schemes are constructed. The schemes are mainly based on zero-forcing and null-space beamforming.

Lower and upper boldface letters denote column vectors and matrices, respectively. $\mathbf{X}^H$ and $\mathbf{X}^\dagger$ denote the Hermitian transpose and the pseudo-inverse of $\mathbf{X}$, respectively. The sequence $(\mathbf{x}(1), \mathbf{x}(2), \dots, \mathbf{x}(N))$ is denoted by $\mathbf{x}^N$. Let $h(\mathbf{x})$ denote the differential entropy of a random vector $\mathbf{x}$, and $I(\mathbf{x}; \mathbf{y})$ denote the mutual information between two random vectors $\mathbf{x}$ and $\mathbf{y}$.

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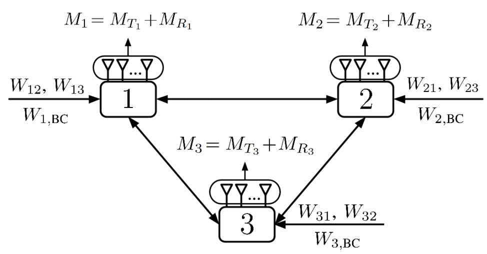

Fig. 1: The system model.

II. System Model

We consider the MIMO 3-node fully-connected interference network, a.k.a. the MIMO 3-way channel, depicted in Fig. 1. Each node has a separate-antenna full-duplex MIMO transceiver where each antenna can be configured for either signal transmission or reception. Consequently, node ℓ, where $\ell \in \mathcal{U} = \{1, 2, 3\}$, has M_ℓ antennas of which it utilizes M_{T_ℓ} antennas for signal transmission and M_{R_ℓ} antennas for signal reception, where $M_{T_\ell} + M_{R_\ell} = M_\ell$. Furthermore, our asymmetric setting entails a different number of antennas at the different nodes. Henceforth, without loss of generality, we assume that $M_1 \geq M_2 \geq M_3$. Moreover, the signals as well as the channel coefficients are assumed to be complex-valued. Similar to [4], [5], we assume that the nodes operate in a perfect full-duplex mode, i.e., each node can transmit and receive messages simultaneously and the effect of residual self-interference, imposed by the transmit antennas on the receive antennas within the same transceiver, is perfectly cancelled or suppressed.

Node i can send two independent unicast messages; W_{ij} and W_{ik} to nodes j and k with rates R_{ij} and R_{ik}, respectively, for $i, j, k \in \mathcal{U}$ and $i \neq j \neq k$. The transmitted signal from node i is denoted by $\mathbf{x}_i \in \mathbb{C}^{M_{T_i} \times 1}$. It is assumed that the power of the transmitted signal from node i is bounded by ρ, i.e., $\mathbb{E}\left\{\|\mathbf{x}_i\|^2\right\} \leq \rho$. Taking into account the aforementioned description of the system model, the received signal at node j at time slot n, denoted by $\mathbf{y}_j(n) \in \mathbb{C}^{M_{R_j} \times 1}$, is given by

$$\mathbf{y}_j(n) = \sum_{i \in \mathcal{U}, i \neq j} \mathbf{H}_{ij}\, \mathbf{x}_i(n) + \mathbf{z}_j(n)\,, \tag{1}$$

where $\mathbf{H}_{ij} \in \mathbb{C}^{M_{R_j} \times M_{T_i}}$ is the random channel matrix from node i to node j, and $\mathbf{z}_j \in \mathbb{C}^{M_{R_j} \times 1}$ is the additive noise signal at node j whose elements are independent and identically distributed (i.i.d.) complex Gaussian random variables with zero mean and unit variance. Throughout this paper, we assume that each node has perfect knowledge of the channel state information (CSI) from the two other nodes. Moreover, for the sake of notational simplicity, we drop the time index n throughout the sequel unless necessary.

Let $\mathbf{y}_\ell^n$ denote the sequence of $\mathbf{y}_\ell$ from time slot 1 up to time slot n, for $\ell \in \mathcal{U}$ and $n \in \mathcal{N} = \{1, 2, \ldots, N\}$. Now, we define the encoder and decoder functions for the considered system model [6]. The encoder function at node i maps its own messages W_{ij} and W_{ik}, and the past values of the received symbols $\mathbf{y}_i^{n-1}$ into the symbol $\mathbf{x}_i(n)$. Therefore, the encoder function $\mathcal{E}_i$ of node i is expressed as

$$\mathbf{x}_i(n) = \mathcal{E}_i\big(W_{ij}, W_{ik}, \mathbf{y}_i^{n-1}\big)\,, \tag{2}$$

where $i, j, k \in \mathcal{U}$ and $i \neq j \neq k$. On the other hand, for a transmission block of length N, the decoder function at node i maps its own messages W_{ij} and W_{ik}, and the received symbols in each block $\mathbf{y}_i^N$ to form estimates of its desired messages $\hat{W}_{ji}$ and $\hat{W}_{ki}$. Therefore, the decoder function $\mathcal{D}_i$ of node i is expressed as

$$\left(\hat{W}_{ji}, \hat{W}_{ki}\right) = \mathcal{D}_i\big(W_{ij}, W_{ik}, \mathbf{y}_i^N\big). \tag{3}$$

In this work, we use the total DoF as the key performance metric to characterize the capacity behavior in the high signal-to-noise ratio (SNR) regime [7]. The DoF of a message W_{ij} with a rate R_{ij} (as a function of the SNR) is designated as d_{ij}, for $i, j \in \mathcal{U}$ and $i \neq j$. It is characterized as

$$d_{ij} = \lim_{\mathrm{SNR} \to \infty} \frac{R_{ij}(\mathrm{SNR})}{\log(\mathrm{SNR})}. \tag{4}$$

The total DoF of the MIMO 3-way channel, d_Σ, is defined as

$$d_\Sigma = d_{12} + d_{13} + d_{21} + d_{23} + d_{31} + d_{32}. \tag{5}$$

III. Main Result

In this section, we characterize the asymmetric total DoF of the full-duplex MIMO 3-way channel. The following theorem presents the main result of this section.

Theorem 1. *The optimal total DoF of the MIMO 3-way channel, with $M_1 \geq M_2 \geq M_3$, where each node sends a unicast message to each of the two other nodes, is given by*

$$d_\Sigma = \min\left\{M_1 + \frac{M_2 + M_3 - M_1}{3},\; M_2 + M_3\right\}. \tag{6}$$

Proof: The converse proof of Theorem 1 is presented in Section III-A, together with the optimal antenna allocation at each node that can achieve the maximum total DoF of the system. Finally, the achievability proof of Theorem 1 is presented in Section III-B.

A. *Converse Proof of Theorem 1*

The proof is divided into three parts. First, the cut-set bounds are provided. Next, the genie-aided bounds are derived. Finally, the optimal antenna allocation at each node is derived in order to maximize the total DoF given by the cut-set and genie-aided bounds.

1) ***Cut-set Bounds****:* The derivation of cut-set bounds hinges on the cut-set theorem [6]. Let $\mathcal{S}$ and $\mathcal{S}^c$ denote the set of source and destination nodes, respectively, where $\mathcal{S}^c$ is the complement of $\mathcal{S}$. We start the proof by arguing that the cooperation of any two nodes among the three nodes does not degrade the DoF [6]. Taking this fact into consideration, we first consider the cut around $\mathcal{S} = \{1\}$ and $\mathcal{S}^c = \{2, 3\}$. This leads to the following inequality

$$d_{12} + d_{13} \leq \min\left\{M_{T_1},\; M_{R_2} + M_{R_3}\right\}. \tag{7}$$

Similarly, the following upper bounds can be obtained

$$d_{21}+d_{23} \leq \min\{M_{T_2},\ M_{R_1}+M_{R_3}\}, \tag{8}$$
$$d_{31}+d_{32} \leq \min\{M_{T_3},\ M_{R_1}+M_{R_2}\}. \tag{9}$$

Adding (7), (8) and (9), we get

$$\begin{aligned} d_\Sigma \leq \min\{&M_{T_1}+M_{T_2}+M_{T_3}, M_{T_1}+M_{T_2}+M_{R_1}+M_{R_2}, \\ &M_{T_1}+M_{T_3}+M_{R_1}+M_{R_3}, M_{T_2}+M_{T_3}+M_{R_2}+M_{R_3}, \\ &M_{T_1}+2M_{R_1}+M_{R_2}+M_{R_3}, M_{T_2}+M_{R_1}+2M_{R_2}+M_{R_3}, \\ &M_{T_3}+M_{R_1}+M_{R_2}+2M_{R_3}, 2(M_{R_1}+M_{R_2}+M_{R_3})\}. \end{aligned} \tag{10}$$

On the other hand, if we consider the cut around $\mathcal{S}=\{1,2\}$ and $\mathcal{S}^c=\{3\}$, we obtain

$$d_{13}+d_{23} \leq \min\{M_{T_1}+M_{T_2},\ M_{R_3}\}. \tag{11}$$

Similarly, the following upper bounds can be obtained

$$d_{21}+d_{31} \leq \min\{M_{T_2}+M_{T_3},\ M_{R_1}\}, \tag{12}$$
$$d_{12}+d_{32} \leq \min\{M_{T_1}+M_{T_3},\ M_{R_2}\}. \tag{13}$$

Adding (11), (12) and (13), we get

$$\begin{aligned} d_\Sigma \leq \min\{&M_{R_1}+M_{R_2}+M_{R_3}, M_{T_1}+M_{T_2}+M_{R_1}+M_{R_2}, \\ &M_{T_1}+M_{T_3}+M_{R_1}+M_{R_3}, M_{T_2}+M_{T_3}+M_{R_2}+M_{R_3}, \\ &M_{R_1}+2M_{T_1}+M_{T_2}+M_{T_3}, M_{R_2}+M_{T_1}+2M_{T_2}+M_{T_3}, \\ &M_{R_3}+M_{T_1}+M_{T_2}+2M_{T_3}, 2(M_{T_1}+M_{T_2}+M_{T_3})\}. \end{aligned} \tag{14}$$

Combining (10) and (14), and then simplifying the resulting expression, the cut-set upper bound on the total DoF of the MIMO 3-way channel is characterized as

$$\begin{aligned} d_\Sigma \leq \min\{&M_{T_2}+M_{T_3}+M_{R_2}+M_{R_3},\ M_{T_1}+M_{T_2}+M_{T_3}, \\ &M_{R_1}+M_{R_2}+M_{R_3}\}. \end{aligned} \tag{15}$$

In cut-set bounds, it is assumed that the nodes on the same side of the cut are fully cooperating. For instance, if we consider the cut around $\mathcal{S}=\{1\}$ and $\mathcal{S}^c=\{2,3\}$, we can imagine a genie that transfers W_{23} to node 3 and W_{32} to node 2. That is why, the cut-set bounds are referred to as the two-sided genie-aided bounds [8]. In order to establish tighter bounds on the total DoF, we resort to the one-sided genie-aided bounds [5], [8], [9] which we refer to as the genie-aided bounds in the sequel.

*2) **Genie-aided Bounds:*** The key idea of genie-aided bounds is that we assume the genie transfers the side-information from one node to another and not the other way around [8]. For example, in cut-set bounds, the genie transfers W_{23} and W_{32} to nodes 3 and 2, respectively. However, in genie-aided bounds, we assume that the genie transfers either W_{23} or W_{32} and, hence, the other message is not known at its respective node a priori.

We assume that every node can decode its desired unicast messages from the other nodes, according to the decoding function in (3), with an arbitrarily small probability of error. For example, node 1 decodes W_{21} and W_{31} using its received signal, $\mathbf{y}_1^N$, and its unicast messages, W_{12} and W_{13}, intended to node 2 and node 3, respectively. Thus, node 1 knows $\mathbf{y}_1^N$, W_{12}, W_{13}, W_{21} and W_{31} after the decoding process. Node 1 cannot decode more messages without being provided with additional side-information. In order to decode more messages, node 1 should be more knowledgeable than some other nodes. Suppose we want node 1 to be able to decode W_{32}. Knowing W_{21}, we should provide node 1 with W_{23} and $\mathbf{y}_2^N$ in order to decode W_{32}. Assume that the genie transfers W_{23} to node 1 as side-information. Then, what is left is to specifically know the additional side-information that is required to be transferred by the genie in order to generate $\mathbf{y}_2^N$. We will elaborate this as follows. Having W_{21} and W_{23}, node 1 can generate $\mathbf{x}_2(1)$. We then evaluate the following expression

$$\begin{aligned} \mathbf{y}_1(1)-\mathbf{H}_{21}\mathbf{x}_2(1) &= \mathbf{H}_{21}\mathbf{x}_2(1)+\mathbf{H}_{31}\mathbf{x}_3(1)+\mathbf{z}_1(1)-\mathbf{H}_{21}\mathbf{x}_2(1) \\ &= \mathbf{H}_{31}\mathbf{x}_3(1)+\mathbf{z}_1(1). \end{aligned} \tag{16}$$

Next, we multiply the previous expression by $\mathbf{H}_{31}^\dagger$ to get

$$\mathbf{H}_{31}^\dagger\left(\mathbf{y}_1(1)-\mathbf{H}_{21}\mathbf{x}_2(1)\right) = \mathbf{x}_3(1)+\mathbf{H}_{31}^\dagger\mathbf{z}_1(1). \tag{17}$$

It is worth mentioning that the left pseudo-inverse of $\mathbf{H}_{31}$ is guaranteed to exist almost surely if and only if $M_{R_1} \geq M_{T_3}$. Let us assume that this condition holds true for now and then we will later study the case when this condition is not satisfied. Taking into consideration Eq. (1), node 1 generates $\mathbf{y}_2(1)$ as follows.

$$\begin{aligned} &\mathbf{H}_{32}\left(\mathbf{x}_3(1)+\mathbf{H}_{31}^\dagger\mathbf{z}_1(1)\right)+\mathbf{H}_{12}\mathbf{x}_1(1) \\ &= \left(\mathbf{H}_{12}\mathbf{x}_1(1)+\mathbf{H}_{32}\mathbf{x}_3(1)+\mathbf{z}_2(1)\right)+\left(\mathbf{H}_{32}\mathbf{H}_{31}^\dagger\mathbf{z}_1(1)-\mathbf{z}_2(1)\right) \\ &= \mathbf{y}_2(1)+\mathbf{g}_{1,W_{23}}(1), \end{aligned} \tag{18}$$

where $\mathbf{g}_{1,W_{23}}(1) = \mathbf{H}_{32}\mathbf{H}_{31}^\dagger\mathbf{z}_1(1)-\mathbf{z}_2(1)$. We can see from (18) that the side-information that node 1 requires is $\mathbf{g}_{1,W_{23}}(1)$ and, hence, node 1 can subtract it from $\mathbf{H}_{23}\left(\mathbf{x}_3(1)+\mathbf{H}_{31}^\dagger\mathbf{z}_1(1)\right)+\mathbf{H}_{21}\mathbf{x}_1(1)$ to generate $\mathbf{y}_2(1)$. Having $\mathbf{y}_2(1)$, W_{21} and W_{23}, node 1 can generate $\mathbf{x}_2(2)$, according to the encoding function in (2). Following the same line of thought explained above, node 1 can accordingly generate $\mathbf{y}_2(2)$. Node 1 reiterates this procedure until it completely generates $\mathbf{y}_2^N$.

To sum up, when the genie transfers W_{23} as well as $\mathbf{g}_{1,W_{23}}^N$ to node 1 as side-information, it becomes more knowledgeable than node 2, that only has W_{21}, W_{23} and $\mathbf{y}_2^N$. Hence, node 1 can decode W_{32} in addition to W_{21} and W_{31}. From Fano's inequality, we can write

$$\begin{aligned} &N\left(R_{21}+R_{31}+R_{32}\right) \\ &\leq I\left(\underbrace{W_{21},W_{31},W_{32}}_{W_1};\ \mathbf{y}_1^N,\underbrace{W_{12},W_{13},W_{23}}_{W_2},\mathbf{g}_{1,W_{23}}^N\right)+N\epsilon_N \\ &= I\left(W_1;\ \mathbf{y}_1^N,W_2,\mathbf{g}_{1,W_{23}}^N\right)+N\epsilon_N \\ &\overset{(a)}{=} I\left(W_1;\ W_2,\mathbf{g}_{1,W_{23}}^N\right)+I\left(W_1;\ \mathbf{y}_1^N \mid W_2,\mathbf{g}_{1,W_{23}}^N\right)+N\epsilon_N \\ &\overset{(b)}{=} I\left(W_1;\ \mathbf{y}_1^N \mid W_2,\mathbf{g}_{1,W_{23}}^N\right)+N\epsilon_N \\ &= h\left(\mathbf{y}_1^N \mid W_2,\mathbf{g}_{1,W_{23}}^N\right)-h\left(\mathbf{y}_1^N \mid W_1,W_2,\mathbf{g}_{1,W_{23}}^N\right)+N\epsilon_N \end{aligned}$$

$$
\begin{aligned}
&\overset{(c)}{\leq} h\left(\mathbf{y}_1^N\right) - h\left(\mathbf{y}_1^N \mid W_1, W_2, \mathbf{g}_{1,W_{23}}^N\right) + N\epsilon_N \\
&\overset{(d)}{=} h\left(\mathbf{y}_1^N\right) - \sum_{n=1}^{N} h\left(\mathbf{y}_1(n) \mid \mathbf{y}_1^{n-1}, W_1, W_2, \mathbf{g}_{1,W_{23}}^N\right) + N\epsilon_N \\
&\leq h\big(\mathbf{y}_1^N\big) - \sum_{n=1}^{N} h\big(\mathbf{y}_1(n) \mid \mathbf{y}_1^{n-1}, W_1, W_2, \mathbf{g}_{1,W_{23}}^N, \ldots \\
&\qquad\qquad \mathbf{y}_2^{n-1}, \mathbf{y}_3^{n-1}\big) + N\epsilon_N \\
&\overset{(e)}{=} h\big(\mathbf{y}_1^N\big) - \sum_{n=1}^{N} h\big(\mathbf{y}_1(n) \mid \mathbf{y}_1^{n-1}, W_1, W_2, \mathbf{g}_{1,W_{23}}^N, \ldots \\
&\qquad\qquad \mathbf{y}_2^{n-1}, \mathbf{y}_3^{n-1}, \mathbf{x}_2^n, \mathbf{x}_3^n, \mathbf{z}_1^{n-1}\big) + N\epsilon_N \\
&\overset{(f)}{=} h\big(\mathbf{y}_1^N\big) - \sum_{n=1}^{N} h\big(\mathbf{z}_1(n) \mid \mathbf{y}_1^{n-1}, W_1, W_2, \mathbf{g}_{1,W_{23}}^N, \ldots \\
&\qquad\qquad \mathbf{y}_2^{n-1}, \mathbf{y}_3^{n-1}, \mathbf{x}_2^n, \mathbf{x}_3^n, \mathbf{z}_1^{n-1}\big) + N\epsilon_N \\
&\overset{(g)}{=} h\big(\mathbf{y}_1^N\big) - \sum_{n=1}^{N} h\big(\mathbf{z}_1(n) \mid \mathbf{g}_{1,W_{23}}^N, \mathbf{z}_1^{n-1}\big) + N\epsilon_N \\
&= h\big(\mathbf{y}_1^N\big) - h\big(\mathbf{z}_1^n \mid \mathbf{g}_{1,W_{23}}^N\big) + N\epsilon_N \\
&\leq \sum_{n=1}^{N} h\left([\,\mathbf{H}_{21}\mathbf{H}_{31}]\begin{bmatrix}\mathbf{x}_2(n)\\ \mathbf{x}_3(n)\end{bmatrix} + \mathbf{z}_1(n)\right) + \mathcal{O}(1) + N\epsilon_N, \quad (19)
\end{aligned}
$$

where $\mathcal{O}(1)$ is a term that is irrelevant to the DoF characterization, (a) follows from the chain rule for mutual information, (b) follows from the fact that W_1, W_2 and $\mathbf{g}_{1,W_{23}}^N$ are independent from each other and, hence, $I\left(W_1;\, W_2, \mathbf{g}_{1,W_{23}}^N\right) = 0$, (c) follows from the fact that conditioning reduces entropy, (d) follows from the chain rule for entropy, (e) follows from the fact that $\mathbf{x}_i(n)$ is a function of W_{ij}, W_{ik} and $\mathbf{y}_i^{n-1}$ for $i,j,k \in \mathcal{U}$ and $i \neq j \neq k$, and $\mathbf{z}_1(n) = \mathbf{y}_1(n) - (\mathbf{H}_{21}\,\mathbf{x}_2(n) + \mathbf{H}_{31}\,\mathbf{x}_3(n))$, (f) follows from the fact that $h\left(\mathbf{H}_{21}\,\mathbf{x}_2(n) + \mathbf{H}_{31}\,\mathbf{x}_3(n) + \mathbf{z}_1(n) \mid \mathbf{x}_2(n), \mathbf{x}_3(3)\right) = h\left(\mathbf{z}_1(n) \mid \mathbf{x}_2(n), \mathbf{x}_3(3)\right)$, (g) follows from the fact that $\mathbf{z}_1(n)$ and $\left\{\mathbf{y}_i^{n-1}, W_1, W_2, \mathbf{x}_j^n\right\}$ are independent, for $i \in \mathcal{U}$ and $j \in \mathcal{U}\backslash\{1\}$. It should be noted that $\epsilon_N \to 0$ as $N \to \infty$. Thus, when $M_{R_1} \geq M_{T_3}$, the total DoF of W_{21}, W_{31} and W_{32} is upper bounded by

$$
\begin{aligned}
N\left(d_{21}+d_{31}+d_{32}\right) &\leq N\left(\operatorname{rank}\left([\,\mathbf{H}_{21}\mathbf{H}_{31}\,]\right)+\epsilon_N\right) \\
&= N\left(\min\{M_{R_1}, M_{T_2}+M_{T_3}\}+\epsilon_N\right) \quad (20)
\end{aligned}
$$

When dividing both sides by N and then letting $N \to \infty$, we obtain

$$
d_{21}+d_{31}+d_{32} \leq \min\{M_{R_1}, M_{T_2}+M_{T_3}\}, \text{ if } M_{R_1} \geq M_{T_3}. \quad (21)
$$

On the other hand, when $M_{T_3} \geq M_{R_1}$, the left pseudo-inverse of $\mathbf{H}_{31}$ does not exist. To tackle this problem, we deduce an upper bound on the total DoF by increasing the number of receive antennas at node 1 such that $M_{R_1} = M_{T_3}$. As a result, the total DoF of W_{21}, W_{31} and W_{32} is upper bounded by

$$
d_{21}+d_{31}+d_{32} \leq \min\{M_{T_3}, M_{T_2}+M_{T_3}\}, \text{ if } M_{T_3} \geq M_{R_1}. \quad (22)
$$

Combining (21) and (22), we finally get

$$
d_{21}+d_{31}+d_{32} \leq \min\left\{\max\left\{M_{R_1}, M_{T_3}\right\}, M_{T_2}+M_{T_3}\right\}. \quad (23)
$$

We have based our previous discussion on the assumption that the genie provides node 1 with W_{23} and $\mathbf{g}_{1,W_{23}}^N$ to be able to decode W_{32}. Now we assume that the genie transfers W_{32} and $\mathbf{g}_{1,W_{32}}^N$ to node 1 in order to decode W_{23}. Following the same approach, we can find that

$$
\mathbf{g}_{1,W_{32}}^N = \mathbf{H}_{23}\mathbf{H}_{21}^{\dagger}\mathbf{z}_1^N - \mathbf{z}_3^N. \quad (24)
$$

Therefore, the total DoF of W_{21}, W_{31} and W_{23} is upper bounded by

$$
d_{21}+d_{31}+d_{23} \leq \min\left\{\max\left\{M_{R_1}, M_{T_2}\right\}, M_{T_2}+M_{T_3}\right\}. \quad (25)
$$

Following the same procedure for deriving the genie-aided bounds from node 1 perspective, we can derive those from node 2 and node 3 perspectives. Combining (23), (25) and the bounds from node 2 and node 3 perspectives, with the cut-set bounds given by (15), the total DoF of the MIMO 3-way channel is upper bounded by

$$
\begin{aligned}
d_{\Sigma} \leq \min\{&M_{T_1}+M_{T_2}+M_{T_3},\ M_{R_1}+M_{R_2}+M_{R_3}, \\
&\max\{M_{R_2}, M_{T_3}\}+\max\{M_{R_3}, M_{T_2}\}, \\
&\max\{M_{R_2}, M_{T_1}\}+\max\{M_{R_1}, M_{T_2}\}, \\
&\max\{M_{R_3}, M_{T_1}\}+\max\{M_{R_1}, M_{T_3}\}\}. \quad (26)
\end{aligned}
$$

The derivation details of (26) are given in [10, Section III-A].

Corollary 1. *The special case of $M_{T_1} = M_{T_2} = M_{T_3} = M_T$ and $M_{R_1} = M_{R_2} = M_{R_3} = M_R$, studied by Maier* et al. *in [5], is covered by* (26)*. In this case, the total DoF of the symmetric MIMO 3-way channel is upper bounded by*

$$
d_{\Sigma} \leq \begin{cases} \min\{3M_R, 2M_T\} & \text{for } M_T \geq M_R, \\ \min\{3M_T, 2M_R\} & \text{for } M_T \leq M_R. \end{cases} \quad (27)
$$

*3) **Optimal Antenna Allocation**:* In this part, we seek the optimal allocation of transmit and receive antennas at each node in terms of M_1, M_2 and M_3 to maximize the upper bound on the total DoF of the MIMO 3-way channel, given by (26). The optimization problem is formulated as follows

$$
\begin{aligned}
\textbf{P1:} \quad &\max_{d_{\Sigma}, M_{T_\ell}, M_{R_\ell}} \ d_{\Sigma} \\
&\text{s.t.} \quad (26), \\
&\qquad M_{T_\ell}+M_{R_\ell} = M_\ell, \ \text{for } \ell \in \{1,2,3\}. \quad (28)
\end{aligned}
$$

Lemma 1. *The total DoF of the MIMO 3-way channel is upper bounded by $d_{\Sigma} \leq d_{\Sigma}^{\star}$, where $d_{\Sigma}^{\star}$ is the optimal solution of **P1**, which is given by*

$$
d_{\Sigma}^{\star} = \begin{cases} M_1 + \dfrac{M_2+M_3-M_1}{3} & \text{for } M_1 \leq M_2+M_3 \\ M_2+M_3 & \text{for } M_1 \geq M_2+M_3. \end{cases} \quad (29)
$$

When $M_1 \leq M_2+M_3$, one optimal antenna allocation that achieves the corresponding maximum total DoF is

$$
\left[M_{R_1}^{\star}, M_{R_2}^{\star}, M_{R_3}^{\star}\right] = \left[0, \frac{M_1+2M_2-M_3}{3}, \frac{M_1+2M_3-M_2}{3}\right]. \quad (30)
$$

On the other hand, when $M_1 \geq M_2+M_3$, one optimal antenna allocation that yields the maximum total DoF in this case is

$$
\left[M_{R_1}^{\star}, M_{R_2}^{\star}, M_{R_3}^{\star}\right] = [M_2+M_3,\ 0,\ 0]. \quad (31)
$$

*Note that $M^\star_{T_\ell} = M_\ell - M^\star_{R_\ell}$ according to the second constraint of **P1**.*

Proof: The details of the solution of **P1** are reported in [10, Appendix A]. This completes the converse proof of Theorem 1. ■

B. Achievability Proof of Theorem 1

In this subsection, we provide the achievable schemes of total DoF of the MIMO 3-way channel described in Theorem 1. Let $i, j, k \in \mathcal{U}$ and $i \neq j \neq k$. A message W_{ij} is encoded at the transmitter into the symbol $\mathbf{u}_{ij} \in \mathbb{C}^{r_{ij}\times 1}$, where $r_{ij} \leq M_{T_i}$. The transmitted signal from node i, $\mathbf{x}_i \in \mathbb{C}^{M_{T_i}\times 1}$, is defined as

$$\mathbf{x}_i = \mathbf{T}_{ij}\mathbf{u}_{ij} + \mathbf{T}_{ik}\mathbf{u}_{ik}, \tag{32}$$

where $\mathbf{T}_{ij} \in \mathbb{C}^{M_{T_i}\times r_{ij}}$ is the precoding matrix for the signal transmitted from node i to node j.

1) $\boldsymbol{M_1 \leq M_2 + M_3}$*:* In this case, the total DoF of the MIMO 3-way channel is bounded by $d_\Sigma \leq M_1 + \frac{M_2+M_3-M_1}{3}$. The transmit and receive antennas at each node are allocated as per (30). It should be noted that if M_{T_ℓ} and M_{R_ℓ}, for $\ell \in \mathcal{U}$, are not integers, we use the symbol extension method over multiple time slots [7]. Then, we proceed with the design of the transmit strategy as explained below. The transmitted signals from each node are

$$\mathbf{x}_1 = \mathbf{T}_{12}\mathbf{u}_{12} + \mathbf{T}_{13}\mathbf{u}_{13},\ \ \mathbf{x}_2 = \mathbf{T}_{23}\mathbf{u}_{23},\ \ \mathbf{x}_3 = \mathbf{T}_{32}\mathbf{u}_{32}, \tag{33}$$

where the dimensions of encoded data symbols $\mathbf{u}_{12}$, $\mathbf{u}_{13}$, $\mathbf{u}_{23}$ and $\mathbf{u}_{32}$ are $(M_{T_1} - M_{R_3})\times 1$, $(M_{T_1} - M_{R_2})\times 1$, $M_{T_2}\times 1$ and $M_{T_3}\times 1$, respectively, whereas the dimensions of precoding matrices $\mathbf{T}_{12}$, $\mathbf{T}_{13}$, $\mathbf{T}_{23}$ and $\mathbf{T}_{32}$ are $M_{T_1} \times (M_{T_1} - M_{R_3})$, $M_{T_1} \times (M_{T_1} - M_{R_2})$, $M_{T_2} \times M_{T_2}$ and $M_{T_3} \times M_{T_3}$, respectively. Note that $\mathbf{T}_{21} = \mathbf{T}_{31} = \mathbf{0}$ since $M_{R_1} = 0$. The precoding matrices $\mathbf{T}_{12}$ and $\mathbf{T}_{13}$ are designed such that

$$\mathbf{T}_{12} \in \text{null}\left(\mathbf{H}_{13}\right),\ \ \mathbf{T}_{13} \in \text{null}\left(\mathbf{H}_{12}\right). \tag{34}$$

It is worth mentioning that the right pseudo-inverses of $\mathbf{H}_{13}$ and $\mathbf{H}_{12}$ exist almost surely owing to the fact that $M_{R_3} \leq M_{T_1}$ and $M_{R_2} \leq M_{T_1}$, respectively. On the other hand, the precoding matrices $\mathbf{T}_{23}$ and $\mathbf{T}_{32}$ are randomly selected. Consequently, the received signals at nodes 2 and 3 are

$$\begin{aligned} \mathbf{y}_2 &= \mathbf{H}_{12}\mathbf{T}_{12}\mathbf{u}_{12} + \mathbf{H}_{32}\mathbf{T}_{32}\mathbf{u}_{32} + \mathbf{z}_2, \\ \mathbf{y}_3 &= \mathbf{H}_{13}\mathbf{T}_{13}\mathbf{u}_{13} + \mathbf{H}_{23}\mathbf{T}_{23}\mathbf{u}_{23} + \mathbf{z}_3. \end{aligned} \tag{35}$$

Node 2 can decode $\mathbf{u}_{12}$ and $\mathbf{u}_{32}$ by projecting $\mathbf{y}_2$ to the null spaces of $\left(\mathbf{H}_{32}\mathbf{T}_{32}\right)^H$ and $\left(\mathbf{H}_{12}\mathbf{T}_{12}\right)^H$, respectively. Let $\mathbf{Q}_{12} \in \mathbb{C}^{M_{R_2}\times\left(M_{R_2}-M_{T_3}\right)}$ and $\mathbf{Q}_{32} \in \mathbb{C}^{M_{R_2}\times\left(M_{R_2}+M_{R_3}-M_{T_1}\right)}$ denote the projection matrices designed by node 2 such that

$$\mathbf{Q}_{12} \in \text{null}\left(\left(\mathbf{H}_{32}\mathbf{T}_{32}\right)^H\right),\ \ \mathbf{Q}_{32} \in \text{null}\left(\left(\mathbf{H}_{12}\mathbf{T}_{12}\right)^H\right). \tag{36}$$

Since we assume that the nodes have prefect CSI knowledge, the zero-forcing estimates of $\mathbf{u}_{12}$ and $\mathbf{u}_{32}$ at node 2 are

$$\begin{aligned} \hat{\mathbf{u}}_{12} &= \mathbf{G}_{12}\left(\mathbf{Q}_{12}^H\mathbf{H}_{12}\mathbf{T}_{12}\mathbf{u}_{12} + \mathbf{Q}_{12}^H\mathbf{z}_2\right), \\ \hat{\mathbf{u}}_{32} &= \mathbf{G}_{32}\left(\mathbf{Q}_{32}^H\mathbf{H}_{32}\mathbf{T}_{32}\mathbf{u}_{32} + \mathbf{Q}_{32}^H\mathbf{z}_3\right), \end{aligned} \tag{37}$$

where $\mathbf{G}_{12} \in \mathbb{C}^{\left(M_{T_1}-M_{R_3}\right)\times\left(M_{T_1}-M_{R_3}\right)}$ and $\mathbf{G}_{32} \in \mathbb{C}^{M_{T_3}\times M_{T_3}}$ are the inverses of $\mathbf{Q}_{12}^H\mathbf{H}_{12}\mathbf{T}_{12}$ and $\mathbf{Q}_{32}^H\mathbf{H}_{32}\mathbf{T}_{32}$, respectively. $\mathbf{G}_{12}$ and $\mathbf{G}_{32}$ are full rank almost surely because $\mathbf{Q}_{12}$ and $\mathbf{Q}_{32}$ are designed independently of $\mathbf{H}_{12}$ and $\mathbf{H}_{32}$, respectively, and $\mathbf{H}_{12}$ and $\mathbf{H}_{32}$ are drawn from a continuous random distribution. Similarly, node 3 can decode $\mathbf{u}_{13}$ and $\mathbf{u}_{23}$. As a result, node 2 decodes $M_{T_1} + M_{T_3} - M_{R_3}$ linearly independent information symbols while node 3 decodes $M_{T_1} + M_{T_2} - M_{R_2}$ linearly independent information symbols. Thus, the scheme achieves a total of $2M_{T_1} + M_{T_2} + M_{T_3} - M_{R_2} - M_{R_3} = M_1 + \frac{M_2+M_3-M_1}{3}$ DoF for $M_1 \leq M_2 + M_3$.

2) $\boldsymbol{M_1 \geq M_2 + M_3}$*:* In this case, the total DoF of the MIMO 3-way channel is bounded by $d_\Sigma \leq M_2 + M_3$. The transmit and receive antennas at each node are allocated as per (31). The transmitted signals from nodes 2 and 3 are

$$\mathbf{x}_2 = \mathbf{T}_{21}\mathbf{u}_{21},\ \ \mathbf{x}_3 = \mathbf{T}_{31}\mathbf{u}_{31}, \tag{38}$$

where $\mathbf{u}_{21} \in \mathbb{C}^{M_{T_2}\times 1}$ and $\mathbf{u}_{31} \in \mathbb{C}^{M_{T_3}\times 1}$, whereas $\mathbf{T}_{21} \in \mathbb{C}^{M_{T_2}\times M_{T_2}}$ and $\mathbf{T}_{31} \in \mathbb{C}^{M_{T_3}\times M_{T_3}}$. The precoding matrices $\mathbf{T}_{21}$ and $\mathbf{T}_{31}$ are randomly selected. The received signal at node 1 is

$$\mathbf{y}_1 = \mathbf{H}_{21}\mathbf{T}_{21}\mathbf{u}_{21} + \mathbf{H}_{31}\mathbf{T}_{31}\mathbf{u}_{31} + \mathbf{z}_1. \tag{39}$$

Analogous to the previous case, node 1 applies zero-forcing to decode $\mathbf{u}_{21}$ and $\mathbf{u}_{31}$ separately. In other words, node 1 can decode $\mathbf{u}_{21}$ and $\mathbf{u}_{31}$ by designing $\mathbf{V}_{21}$ and $\mathbf{V}_{31}$ such that $\mathbf{V}_{21} \in \text{null}\left(\left(\mathbf{H}_{31}\mathbf{T}_{31}\right)^H\right)$ and $\mathbf{V}_{31} \in \text{null}\left(\left(\mathbf{H}_{21}\mathbf{T}_{21}\right)^H\right)$, respectively. Afterwards, the zero-forcing estimates of $\mathbf{u}_{21}$ and $\mathbf{u}_{31}$ are obtained via evaluating the expressions $\mathbf{V}_{21}^H\mathbf{y}_1$ and $\mathbf{V}_{31}^H\mathbf{y}_1$, respectively. As a result, node 1 decodes a total of $M_{T_2} + M_{T_3}$ independent information symbols are decoded and, hence, the scheme achieves $M_2 + M_3$ DoF for $M_1 \geq M_2 + M_3$. This completes the achievability proof of Theorem 1. ■

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