Differential Unitary Space-Time Constellations from Spherical Codes

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***Abstract*—We construct a new class of unitary space-time constellations that is suitable for multi-antenna differential signaling. This class of constellations follows a certain structure that is inspired by the so-called biorthogonal spherical codes. As a consequence of this, we show that the proposed class achieves optimal diversity sum in some cases and close-to-optimal diversity sum in the remaining cases. Furthermore, we demonstrate how such constellations are tuned in order to improve the diversity product while maintaining the same diversity sum, thereby yielding constellations with favorable diversity sum and product. Numerical evaluations reveal that the proposed construction achieves close or better performance than the best-known constellations over a wide range of signal-to-noise ratios of typical operation, with the added advantage that it can be generated for any number of transmit antennas.**

***Index Terms*—Differential unitary space-time constellations, diversity sum, diversity product, spherical codes.**

I. Introduction

Multi-antenna systems have established themselves as a reliable means to effectively increase the rate at which information is transmitted. A significant portion of the literary work in this area has been devoted to the so-called coherent mode of operation. In this paradigm, channel state information (CSI) is assumed to be known at the receiver [1]. Contrary to coherent multi-antenna systems are those systems in which no knowledge of CSI is required. The use of non-coherent techniques has been particularly attractive in scenarios where learning CSI is either costly or impractical. Recently, non-coherent MIMO techniques have regained a lot of interest motivated by the evolution of applications with nodes that have limited power and/or limited processing capabilities [2]–[4] (the interested reader is referred to [5] and references therein).

A line of work initiated in [6] introduces a multi-antenna paradigm known as Differential Unitary Space-Time (DUST) Coding when channel coefficients do not change dramatically over consecutive transmission blocks. In this model, the transmitted information is differentially encoded into a sequence of unitary matrices, where the transmitted matrix in the current block transmission is determined by the transmitted matrix in the previous block as well as the current transmitted symbol. The design criteria of DUST constellations is generally guided by two important measures [7]. The diversity sum (DS) is the main performance measure at low signal-to-noise ratios (SNRs), whereas the diversity product (DP) is the main performance measure at high SNRs. With a particular interest in high-SNR regime, the majority of existing designs have focused solely on maximizing the DP (e.g., [6]–[9]). However, the authors in [10] have shown that constellations with higher DS and slightly smaller DP can still outperform those that purely maximize the DP over a wide range of SNRs of interest. This is because the effect of superior DP may not kick in until after the range of SNRs of typical operation.

Motivated by the analysis in [10], in this letter, we seek to construct a family of constellations with good DS and DP. In particular, we propose a (suboptimal) two-step approach to the design of DUST constellations. In the first step, constellations with good DSs are constructed. Particularly, the structure of such constellations will follow the geometric structure of the so-called biorthogonal codes [11], [12], a family of spherical codes with specific sizes and whose packing radius is optimal. The resulting constellations will be referred to as the Biorthogonal DUST (Bi-DUST) constellations. In the second step, the elements of Bi-DUST constellations will be subsequently tuned in order to maximize the DP. We will show that the flexible structure of the Bi-DUST constellations allows for such tuning without impacting the DS. In other words, the modified Bi-DUST (MBi-DUST) constellation will possess good DS and DP. The main contributions of this letter are as follows:

- We establish a connection between spherical codes and DUST constellations. By exploiting this connection, we construct a new family of DUST constellations (i.e., Bi-DUST) inspired geometrically by biorthogonal spherical codes. This family achieves optimal DS over a wide range of constellation sizes and for *any number* of transmit antennas.
- We propose a systematic, optimization-based approach to enhance the DP of the proposed Bi-DUST while maintaining its DS performance. Contrary to existing designs (e.g., [8], [10]), the underlying optimization is performed using grid search over a fixed number of variables that does not scale with the number of antennas, thereby reducing the construction complexity of our design significantly. The new family of codes is termed MBi-DUST.
- We present performance comparisons against some of the best-known constellations and demonstrate how the proposed family improves on some of the existing results.

The rest of this letter is organized as follows. In Section II, the system model and some preliminaries are presented. In Section III, we present the design details of our proposed family of DUST MIMO codes. In Section IV, some simulation results are presented and Section V concludes the paper

II. System Model and Preliminaries

A. System Model

We consider the Rayleigh flat-fading communication scenario in [7]. In this model, the channel coefficients are constant over a block transmission of T symbol durations. Using M and N to denote the number of transmit and receive antennas, the $T \times N$ received signal in the τ-th block can be described by

$$\boldsymbol{X}_\tau = \sqrt{\rho}\boldsymbol{S}_\tau \boldsymbol{H}_\tau + \boldsymbol{W}_\tau, \qquad \tau = 1, 2, \ldots, \tag{1}$$

where $\boldsymbol{S}_\tau$ is the $T \times M$ transmitted signal, $\boldsymbol{H}_\tau$ is the $M \times N$ channel matrix, whose coefficients are unknown to both the transmitter and receiver, $\boldsymbol{W}_\tau$ is the $T \times N$ noise matrix, and ρ is the SNR. The entries of $\boldsymbol{H}_\tau$ and $\boldsymbol{W}_\tau$ are independent and identically distributed (i.i.d.) complex Gaussian $\mathcal{CN}(0,1)$. Additionally, it is further assumed that the channel coefficients follow a slow-fading model such that $\boldsymbol{H}_\tau \approx \boldsymbol{H}_{\tau-1}$ holds.

In DUST coding, $T = M$ is assumed. In this case, the transmitter selects one of L possible $M \times M$ unitary matrices from the set $\mathcal{V} = \{\boldsymbol{V}_1, \ldots, \boldsymbol{V}_L\}$. In the τ-th block, the chosen matrix $\boldsymbol{V}_{z_\tau} \in \mathcal{V}$ is encoded into the transmitted signal according to the fundamental differential encoding equation [6]

$$\boldsymbol{S}_\tau = \boldsymbol{V}_{z_\tau} \boldsymbol{S}_{\tau-1} \qquad \tau = 1, 2, \ldots, \tag{2}$$

with $\boldsymbol{S}_0 = \boldsymbol{I}_M$. Substituting (2) into (1), one can readily arrive at the fundamental differential receiver equations [6]

$$\boldsymbol{X}_\tau = \boldsymbol{V}_{z_\tau} \boldsymbol{X}_{\tau-1} + \sqrt{2}\boldsymbol{W}' \qquad \tau = 1, 2, \ldots, \tag{3}$$

where $\boldsymbol{W}'$ is an $M \times N$ matrix with i.i.d. $\mathcal{CN}(0,1)$ noise entries. At the receiver, the maximum likelihood (ML) decoding is performed according to the following rule

$$\min_{1 \le l \le L} \|\boldsymbol{X}_\tau - \boldsymbol{V}_l \boldsymbol{X}_{\tau-1}\|_F. \tag{4}$$

The pairwise error probability (PEP) of mistaking some $\boldsymbol{V}_i$ for $\boldsymbol{V}_{j \neq i}$ is upper bounded by the following Chernoff bound [7]

$$P_{i,j} \le \frac{1}{2} \prod_{m=1}^{M} \left[1 + \frac{\rho^2 \left[\sigma_m^{(i,j)}\right]^2}{4(1+2\rho)} \right]^{-N}, \tag{5}$$

where $\sigma_m^{(i,j)}$ is the m^{th} singular value of $\boldsymbol{V}_i - \boldsymbol{V}_j$, for $m = 1, \ldots, M$. Specifically, two main metrics may be extracted from this expression (cf., [7]): The DS

$$\sigma(\mathcal{V}) = \frac{1}{2\sqrt{M}} \min_{i \neq j} \left\|\boldsymbol{V}_i - \boldsymbol{V}_j\right\|_F = \frac{1}{2\sqrt{M}} \min_{i \neq j} \left(\sum_{m=1}^{M} \left[\sigma_m^{(i,j)}\right]^2 \right)^{1/2}$$

is the dominant performance metric at low SNRs, whereas the DP

$$\mu(\mathcal{V}) = \frac{1}{2} \min_{i \neq j} \sqrt[M]{\left|\boldsymbol{V}_i - \boldsymbol{V}_j\right|} = \frac{1}{2} \min_{i \neq j} \left(\prod_{m=1}^{M} \left[\sigma_m^{(i,j)}\right]^2 \right)^{\frac{1}{2M}},$$

is the dominant performance metric at high SNRs. As we shall see later, both metrics play an important part in this letter.

B. Spherical codes and their connection with DUST constellations

A spherical code [11] is a finite subset of the $(D-1)$-dimensional sphere in $\mathbb{R}^D$, i.e., it is the set $\xi_D(L) = \left\{\boldsymbol{x}_i \in \mathbb{R}^D \middle| \|\boldsymbol{x}_i\| = 1, i = 1, \ldots, L\right\}$, where $\|\cdot\|$ denotes the standard Euclidean distance. For any spherical code, the packing distance $d(\xi_D(L))$ is the minimum separation between two distinct points in the code

$$d(\xi_D(L)) = \min_{x_i \neq x_j \in \xi_D(L)} \|\boldsymbol{x}_i - \boldsymbol{x}_j\|. \tag{6}$$

For reasons that will become clear later, we are interested in the attainable upper bounds on $d(\xi_D(L))$. In particular, for some D and L, let $\delta(D, L)$ be the maximum packing distance taken over all possible arrangements of $\xi_D(L)$. Additionally, let $\xi_D^{\text{opt}}(L)$ be the corresponding *optimal* arrangement such that $d(\xi_D^{\text{opt}}(L)) = \delta(D, L)$. For general L and D, finding $\delta(D, L)$ and $\xi_D^{\text{opt}}(L)$ is an open problem. However, a partial solution is provided by the following classical result [11, Chapter 1].

Lemma 1: Let $L \ge 2$, $\{\boldsymbol{x}\}^c$ be the set of all cyclic rotations of some vector $\boldsymbol{x}$, and $\boldsymbol{1}_n$ ($\boldsymbol{0}_n$) be the vector of all ones (zeros) of length n, then the following holds:

- *Simplex bound*: $L \le D + 1$, then $\delta(D, L) = \sqrt{\frac{2L}{L-1}}$, with $\xi^{\text{opt}}(L) = \frac{1}{\sqrt{L+L^2}} \left\{ \left[\boldsymbol{1}_L^T, -L\right]^T \right\}^c$.
- *Biorthogonal bound*: $D+1 < L \le 2D$, then $\delta(D, L) = \sqrt{2}$, and:
  - For $L = 2D$, $\xi_D^{\text{opt}}(L) = \xi_D^{\text{opt}}(2D) = \left\{ \left[\pm 1, \boldsymbol{0}_{D-1}^T\right]^T \right\}^c$.
  - For $D+1 < L < 2D$, $\xi_D^{\text{opt}}(L)$ is obtained by deleting any $2D - L$ elements from $\xi_D^{\text{opt}}(2D)$.

Since $\mathbb{C}^{M \times M}$ is isomorphic to $\mathbb{R}^{2M^2}$, a DUST constellation $\mathcal{V} = \{\boldsymbol{V}_1, \ldots, \boldsymbol{V}_L\}$ can be regarded as a constant norm code[1] of size L and dimension $D = 2M^2$. Furthermore, the DS of $\mathcal{V}$ is essentially equivalent to one half the packing distance of the associated spherical code. Aided with this observation, we have just proved the following theorem.

Theorem 1: Let $L > 2$ and $\mathcal{V} = \{\boldsymbol{V}_1, \ldots, \boldsymbol{V}_L\}$ be a DUST constellation, the following bounds hold on $\sigma(\mathcal{V})$

- *Simplex bound*: $L \le 2M^2 + 1$, then $\sigma(\mathcal{V}) \le \sqrt{\frac{L}{2(L-1)}}$.
- *Biorthogonal bound*: $2M^2 + 1 < L \le 4M^2$, then $\sigma(\mathcal{V}) \le \frac{1}{\sqrt{2}}$.
- $L > 4M^2$, then $\sigma(\mathcal{V}) \le \frac{1}{\sqrt{2}}$.

We conclude this section with a number of remarks. First, while the simplex/biorthogonal bounds are tight for spherical codes, it is not clear whether there exist DUST constellations whose DS achieves such bounds. This is due to the additional constraints that the columns of any $V \in \mathcal{V}$ are pairwise orthogonal. However, as it turns out, the biorthogonal bound can always be achieved for DUST constellations. Specifically, in Section III, we introduce a family of DUST constellations, which we call the Bi-DUST family, that always attains the biorthogonal bound. Second, to construct Bi-DUST

[1] A constant norm code is defined a subset of points whose norm is some constant α, i.e, $\alpha\xi_D(L)$. Thus, strictly speaking, a DUST constellation is homomorphic to a constant norm code of factor $\alpha = \sqrt{\operatorname{Tr} \boldsymbol{V}_i \boldsymbol{V}_i^\dagger} = \sqrt{\operatorname{Tr} \boldsymbol{I}_M} = \sqrt{M}$.

constellations with sizes $L \in \{2M^2+2, \ldots, 4M^2\}$, we must construct a Bi-DUST constellations of size $4M^2$. Specifically, it follows from Lemma 1 that any Bi-DUST constellation with $L \in \{2M^2+2, \ldots, 4M^2\}$ can automatically be obtained from the Bi-DUST constellation of size $4M^2$ by deletion of elements. Thus, for the remainder of the paper, we will only construct constellations of sizes $L = 4M^2$. Finally, in contrast to the biorthogonal case, it remains an open question whether the simplex bound is achievable for DUST constellations. Instead of attempting to achieve the simplex bound, we restrict ourselves to the use of the Bi-DUST family and deletion, even when $L \le 2M^2+1$. In this case, we notice that the gap from optimality in the DS can never be greater than $\frac{1}{\sqrt{2}}\left(\sqrt{\frac{L}{L-1}}-1\right)$.

III. Proposed Family of DUST Constellations

As stated in Section I, our proposed approach to the design of DUST constellations follows a two-step process. In the first stage of this process, a Bi-DUST constellation is obtained. Whereas in the second stage, the Bi-DUST constellation is subsequently altered in order to maximize its DP. The details are covered in the next two subsections.

A. Step one: Finding the Bi-DUST constellation

Without any loss of generality, we restrict ourselves to the case where $L = 4M^2$, because, as previously noted, constellations with $L < 4M^2$ can be obtained by simple deletion of some elements and still maintain the biorthgonal bound. The structure of the biorthogonl DUST constellation will follow certain characteristics of the spherical code achieving the biorthogonal bound. In Lemma 1, such spherical code is presented when $L = 2D$. Particularly, in a D-dimensional space, the code elements span D orthogonal directions, as well as their negatives. Motivated by this structure, we propose that the corresponding Bi-DUST constellation $\mathcal{V}$ is composed of $4M^2$, $M \times M$, unitary matrices $\pm\boldsymbol{V}_1, \ldots, \pm\boldsymbol{V}_{2M^2}$, where $\boldsymbol{V}_i$'s are pairwise orthogonal with respect to some inner product[2]. Since we treat elements in $\mathbb{C}^{M\times M}$ as corresponding elements in $\mathbb{R}^{2M^2}$, it remains to define the appropriate inner product that properly respects this isomorphism. Let $\boldsymbol{V}_1, \boldsymbol{V}_2 \in \mathbb{C}^{M\times M}$ be any two matrices. Also, let $f(\boldsymbol{V}_i), i \in \{1,2\}$ be their images in $\mathbb{R}^{2M^2}$ under the vector space isomorphism $f : \mathbb{C}^{M\times M} \to \mathbb{R}^{2M^2}$. We wish to define the inner product $\langle \cdot,\cdot \rangle_{\mathbb{C}^{M\times M}}$ such that $\langle \boldsymbol{V}_1, \boldsymbol{V}_2\rangle_{\mathbb{C}^{M\times M}} = \langle f(\boldsymbol{V}_1), f(\boldsymbol{V}_2)\rangle_{\mathbb{R}^{2M^2}} = f(\boldsymbol{V}_1)^T f(\boldsymbol{V}_2)$. One may see that such inner product is given by:

$$\langle \boldsymbol{V}_1, \boldsymbol{V}_2\rangle = \Re(\mathrm{Tr}\, \boldsymbol{V}_1^\dagger \boldsymbol{V}_2), \tag{7}$$

where $\Re$ denotes the real part. We now proceed to find the matrices $\boldsymbol{V}_1, \ldots, \boldsymbol{V}_{2M^2}$. The following result imposes some structure on such matrices.

Proposition 1: Let $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_{2M}$ be $2M$, $M \times M$, diagonal unitary matrices, where $\boldsymbol{D}_m$ and $\boldsymbol{D}_{n\neq m}$ are pairwise orthogonal with respect to (7). Additionally, let $\boldsymbol{B}$ be the cyclic rotation matrix

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{0}_{M-1} & \boldsymbol{I}_{M-1} \\ 1 & \boldsymbol{0}_{M-1}^T \end{bmatrix}.$$

For $m = 1, \ldots, 2M$, and $n = 0, \ldots, M-1$, the matrices

$$\boldsymbol{V}_{m+2nM} = \boldsymbol{D}_m \boldsymbol{B}^n \tag{8}$$

are pairwise orthogonal with respect to the inner product in (7). Moreover, the DS of $\mathcal{V} = \{\pm\boldsymbol{V}_1, \ldots, \pm\boldsymbol{V}_{2M^2}\}$ attains the biorthogonal bound.

Proof: To prove the Lemma, we consider $\langle \boldsymbol{V}_{m+2nM}, \boldsymbol{V}_{m'+2n'M}\rangle$. We consider two independent cases. If $n = n'$ and $m \neq m'$, this expression simplifies to $\langle \boldsymbol{D}_m, \boldsymbol{D}_{m'}\rangle$, which is zero by our assumption. On the other hand, if $n \neq n'$, it is straightforward to show that $\boldsymbol{V}_{m+2nM}^\dagger \boldsymbol{V}_{m'+2n'M}$ has zero diagonal entries. ■

We remark that the action of left multiplying by $\boldsymbol{B}^n$ amounts to cyclically rotating the columns of the matrix $\boldsymbol{D}_m$ n times. Since, $\boldsymbol{B}^0 = \boldsymbol{I}_M$, the diagonal matrices $\boldsymbol{D}_m$ are always in $\mathcal{V}$. As a consequence of this Proposition, the task of finding $2M^2$, $M \times M$ unitary matrices, is now reduced to a simpler task of finding $2M$, $M \times M$ diagonal unitary matrices that are pairwise orthogonal with respect to (7). We now introduce the proposed $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_{2M}$ and their negatives. We treat the following cases independently.

1) $M = 2^P$, P integer: In this case, we propose the set of matrices $\mathcal{D}_1 = \{\boldsymbol{D}_1, \ldots, \boldsymbol{D}_{4M}\}$ whose elements take on the form of a diagonal cyclic code [6]:

$$\boldsymbol{D}_l = \begin{bmatrix} e^{j(2\pi/4M)u_1} & 0 & \ldots \\ 0 & \ddots & 0 \\ 0 & \ldots & e^{j(2\pi/4M)u_M} \end{bmatrix}^l \tag{9}$$

with $u_m = 2m-1$, $j = \sqrt{-1}$, and $l = 0, \ldots, 4M-1$.

2) $M \neq 2^P$: In this case, we define the following set $\mathcal{D}_2 = \{\boldsymbol{D}_1, \ldots, \boldsymbol{D}_{4M}\}$, with elements

$$\boldsymbol{D}_{l_1+Ml_2} = (j)^{l_2} \begin{bmatrix} e^{j(2\pi/M)u_1} & 0 & \ldots \\ 0 & \ddots & 0 \\ 0 & \ldots & e^{j(2\pi/M)u_M} \end{bmatrix}^{l_1} \tag{10}$$

where $l_1 = 0, \ldots, M-1$, $l_2 = 0,1,2,3$, $j = \sqrt{-1}$, and $u_m = m-1$. For both (9) and (10), it is straightforward to check that $\boldsymbol{D}_1, \ldots \boldsymbol{D}_{2M}$ are pairwise orthogonal with respect to (7) and that $\boldsymbol{D}_{l+2M} = -\boldsymbol{D}_l$. Thus, the matrices $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_{4M}$ compactly describe the $2M$ orthogonal matrices and their negatives. Hence, using Proposition (1), we conclude that the DS of the corresponding constellation attains the biorthogonal bound.

B. Step two: Maximizing the DP of Bi-DUST constellation

Thus far, we have only taken into consideration the DS of the Bi-DUST constellation. In this section, we turn our attention to the DP of such constellations. The following proposition characterizes the DP of Bi-DUST constellations.

Proposition 2: The family of Bi-DUST constellations introduced in Section III-A has DP $\mu(\mathcal{V}) = 0$. ■

It is advantageous at this point to identify which constellation elements are responsible for the vanishing DP. In particular, the vanishing DP is due to: 1) Constellation elements of $\mathcal{V}$

[2] Two matrices $\boldsymbol{V}_m$ and $\boldsymbol{V}_n$ are orthogonal if their inner product vanishes, i.e., $\langle \boldsymbol{V}_m, \boldsymbol{V}_n\rangle = 0$. This is not to be confused with the matrices themselves being unitary, thus having columns that are orthogonal.

which are obtained by cyclically rotating[3] the columns of $\boldsymbol{D}_m$. And, 2) When M is not a power of two, there always exists some u_m, with $\gcd(u_m, M) > 1$,[4] and thus $\mu(\mathcal{D}_2) = 0$ (cf., [6, Section VI-C]). To remedy these issues, we seek to modify the currently-developed constellation in a manner that yields a strictly positive DP but leaves the DS intact. The next Theorem suggests one possible adjustment.

Theorem 2: [MBi-DUST Constellations] Let $L = 4M^2$, $\tilde{\boldsymbol{B}}$ be the $M \times M$ modified cyclic rotation matrix

$$\tilde{\boldsymbol{B}}(\phi) = \begin{bmatrix} \mathbf{0}_{M-1} & \boldsymbol{I}_{M-1} \\ e^{j\phi} & \mathbf{0}_{M-1}^T \end{bmatrix}. \tag{11}$$

Assume that $\mathcal{D}_1 = \{\boldsymbol{D}_1, \ldots, \boldsymbol{D}_{4M}\}$, where $\boldsymbol{D}_l$ is as defined in (9), and $\tilde{\mathcal{D}}_2(\psi) = \{\tilde{\boldsymbol{D}}_1(\psi), \ldots, \tilde{\boldsymbol{D}}_{4M}(\psi)\}$, such that

$$\tilde{\boldsymbol{D}}_{l_1+Ml_2}(\psi) = (j)^{l_2} e^{j\psi l_1} \begin{bmatrix} e^{j(2\pi/M)u_1} & 0 & \cdots \\ 0 & \ddots & 0 \\ 0 & \cdots & e^{j(2\pi/M)u_M} \end{bmatrix}^{l_1}, \tag{12}$$

where $l_1 = 0, \ldots, M-1$, and $l_2 = 0, 1, 2, 3$. Define the constellation $\mathcal{V} = \{\boldsymbol{V}_1, \ldots, \boldsymbol{V}_L\}$, where $\boldsymbol{V}_{m+2nM} = \boldsymbol{D}_m \tilde{\boldsymbol{B}}^n(\phi)$ if $M = 2^P$, and $\boldsymbol{V}_{m+2nM} = \tilde{\boldsymbol{D}}_m(\psi)\tilde{\boldsymbol{B}}^n(\phi)$ if $M \neq 2^P$. Then, $\mathcal{V}$ possesses the following properties:

- If $M = 2^P$, then $\sigma(\mathcal{V}) = 1/\sqrt{2}, \forall \phi \in [0, 2\pi]$.
- If $M = 2^P$, there exists some ϕ for which $\mu(\mathcal{V}) > 0$.
- If $M \neq 2^P$, then $\sigma(\mathcal{V}) = 1/\sqrt{2}, \forall \phi, \psi \in [0, 2\pi]$.
- If $M \neq 2^P$, there exist some ϕ, ψ for which $\mu(\mathcal{V}) > 0$.

Proof: The proof follows from direct calculations. We omit the details. ■

The rationale behind the newly introduced variables ϕ and ψ in (11) and (12) is as follows: First, the variable ϕ is introduced to overcome the vanishing DP issue that arises due to cyclically rotating $\boldsymbol{D}_m$. Particularly, every time a matrix is cyclically rotated a phase shift $e^{j\phi}$ is also applied. Theorem 2 shows that such transformation preserves the DS but can potentially improve the DP. Likewise, the other variable ψ serves a similar purpose for the second issue where $\gcd(u_m, M) > 1$.

In the construction of the MBi-DUST constellations, we choose the variables ϕ and ψ to maximize the DP using a grid search. In particular, we first construct the Bi-DUST constellation according to Section III-A. Next, for $M = 2^P$, we maximize the DP over the choices $\phi \in \{\frac{2\pi\times 0}{M^2L}, \ldots, \frac{2\pi(M^2L-1)}{M^2L}\}$. Likewise, for $M \neq 2^P$, we maximize the DP over the choices $\phi, \psi \in \{\frac{2\pi\times 0}{M^2L}, \ldots, \frac{2\pi(ML-1)}{M^2L}\}^2$. For each choice of angle (or pair of angles), the modified constellation is found and the corresponding DP is computed. The chosen MBi-DUST constellation is the one that maximizes the DP over all considered choices of the angles.

Remark 1: The design of MBi-DUST constellations entails a grid search over only two variables and whose size grows as $O(M^2)$. Hence, the search complexity scales only quadratically in the number of transmit antennas. This is contrary to the exhaustive design of paramteric constellations [8], [9] that requires more variables as the number of transmit antennas is increased, thus making the overall search complexity grow exponentially in the number of antennas.

Remark 2: Unlike the parametric constellations in [8], [9], the MBi-DUST constellations are not restricted to a certain number of transmit antennas. and can be readily constructed for any number of transmit antennas and over a wide range of constellation sizes, namely $2 \leq L < 4M^2$.

Together, these two remarks make the MBi-DUST family particularly attractive when the number transmit of antennas is large (e.g., in massive MIMO [13]).

TABLE I
COMPARISON OF DS AND DP OF MBi-DUST

M	L	MBi-DUST		hi-SNR		lo-SNR		Cyclic	
		σ	μ	σ	μ	σ	μ	σ	μ
2	8	0.707	0.595	N/A	N/A	N/A	N/A	0.707	0.595
	16	0.707	0.595	N/A	N/A	N/A	N/A	0.383	0.383
3	8	0.707	0.457	0.639	0.647	0.707	0.457	0.618	0.513
	16	0.707	0.365	0.626	0.565	0.673	0.508	0.588	0.448
	32	0.707	0.291	0.493	0.478	0.658	0.331	0.383	0.334
4	8	0.707	0.595	0.707	0.707	0.707	0.707	0.707	0.595
	16	0.707	0.545	0.707	0.615	0.707	0.615	0.707	0.545
	32	0.707	0.545	0.707	0.595	0.707	0.595	0.383	0.383
	64	0.707	0.458	0.707	0.437	0.707	0.437	0.421	0.340

IV. NUMERICAL SIMULATIONS

In this section, the performance of the proposed family of MBi-DUST constellations is compared against the diagonal cyclic constellations in [6] and the constellation family in [8]. In an attempt to generalize the so-called parametric codes in [7], the authors in [8] present a class of DUST constellations for three to six antennas. Using computationally-exhaustive numerical search, they were able to construct constellations that maximize either the DS or DP. The corresponding constellations were referred to as lo-SNR and hi-SNR, respectively. We adopt the same naming here when referring to these constellations. For the MBi-DUST family, to obtain a constellations of the desired size $L < 4M^2$, we first construct the MBi-DUST with size $4M^2$. Then, a deletion policy that prioritizes the removal of $\boldsymbol{D}_m\tilde{\boldsymbol{B}}^n$ over $\boldsymbol{D}_m$ is applied. Table I summarizes the DS and DP of our constellations as well as those of constellations in [6], [8]. In all cases, we observe that the MBi-DUST constellations have better/equal DS than/to that of other constellations. Additionally, when $M = 4$ and $L = 64$, the MBi-DUST constellation has the best DS and DP out of all four constellations. In all other cases, however, the DP of the MBi-DUST is strictly smaller than the DP of the corresponding lo-SNR and hi-SNR constellations. Therefore, we expect that at sufficiently high SNRs, the constellations in [8] will eventually exhibit better performance than the MBi-DUST counterpart.

Using the uncoded block error rate (BER) curves, we now seek to quantify this SNR value. In Fig. 1, we plot the BER of the MBi-DUST, lo-SNR, hi-SNR, and cyclic diagonal constellations when $M = 3$, $N = 2$, and for $L = 16$ and $L = 32$. We model the channel as a block, Rayleigh flat-fading channel. Surprisingly, for $L = 32$, we observe that up

[3] It can be readily proven that any permutations of the columns of some $\boldsymbol{D}_m$ will yield a new matrix whose DP with $\boldsymbol{D}_m$ is zero.

[4] For instance, $u_1 = 0$, and thus always shares a common factor with M.

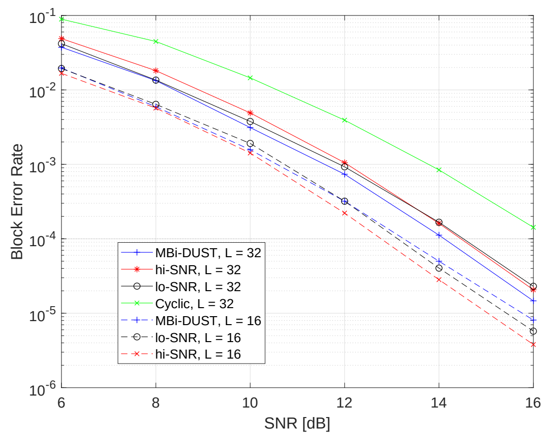

Fig. 1. Performance comparison for $M = 3$, $N = 2$ with different constellation sizes.

TABLE II
DP Spectrum for MBi-DUST and lo-SNR, $M = 3$, $L = 32$.

	MBi-DUST	lo-SNR
$d \leq 0.45$	96	128
$0.45 < d \leq 0.55$	128	96
$0.55 < d \leq 0.65$	224	160
$0.65 < d \leq 0.75$	32	96
$0.75 < d \leq 1$	16	16

to an SNR of 16 dB and $P_e \approx 10^{-5}$, none of the three other constellations outperform the MBi-DUST constellation. This indeed agrees with the analysis in [10] that optimizing the DP alone may not guarantee the best possible performance over the range of SNRs of interest. In fact, we further argue that the lo-SNR constellation may not show any performance improvement over the MBi-DUST one at any SNR. To see this, we refer the reader to Table II in which the DP spectrum of the MBi-DUST is compared with the lo-SNR constellation. Upon inspection of this spectrum, we see that the lo-SNR constellation tends to have denser spectrum at lower values, despite having an overall higher DP.

For completeness, we also compare the BER of those constellations when $M = 3$, $L = 16 < 2M^2 + 1$, ie., when the MBi-DUST constellations no longer achieve the maximum possible DS. In this case, the MBi-DUST constellation is inferior to the constellations [8]. In addition, the performance gap tends to be more noticeable as the SNR is increased, which indicates a limitation in our deletion policy. We remark that all the codes presented above achieve the same diversity order, as all of them have the same BER rate of decay at high SNRs.

Finally, Fig. 2 shows the BER performance for the case of $M = N = 4$ and $L = 64$. In this case, our proposed MBi-DUST code has the same DS value as the lo-SNR and hi-SNR codes while achieving a slightly higher DP value compared to the aforementioned codes. As clear from the figure, all the codes achieve almost the same BER performance with a slight gain for our proposed MBi-DUST code at high SNRs attributed to the fact that it has a slightly higher DP value.

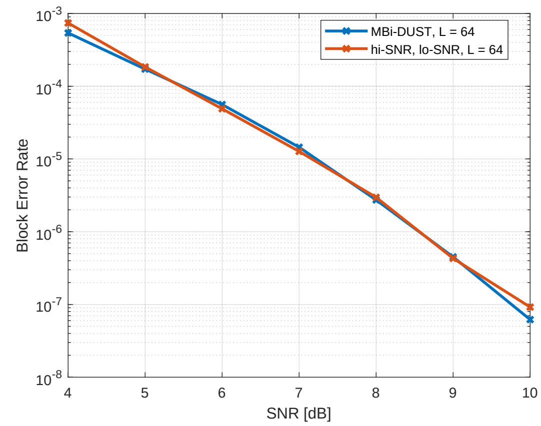

Fig. 2. Performance comparison (BER vs SNR) for $M = 4$, $N = 4$, and $L = 64$, corresponding to a rate of 1.75 bits per channel use.

V. Conclusion

In this letter, we introduced a novel class of unitary constellations that is derived from the classical biorthogonal spherical codes. Under specific conditions, this class was shown to achieve optimal diversity sum. Additionally, its flexible structure allowed for improving the diversity product without disturbing the optimality of the diversity sum and was thus utilized to construct good constellations with respect to both measures. Finally, the proposed construction is comparable to the best-known parametric constellations. However, unlike those constellations, it can be readily obtained for any number of transmit antennas.

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