VIBRATION OF CONTINUOUS SYSTEMS

Introduction

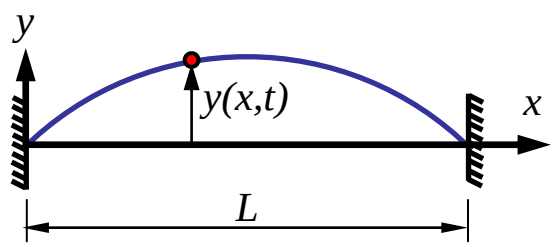
Models of vibratory systems can be divided into two broad classes, *lumped* and *continuous*, depending on the nature of the parameters. In the case of lumped systems, the components are discrete, with the mass assumed to be rigid and concentrated at individual points, and with the stiffness taking the form of massless springs connecting the rigid masses. The masses and springs represent the system parameters, and we refer to such models as *discrete* or *lumped-parameter* models. The motion of discrete systems is governed by *ordinary* differential equations. *Continuous* systems, on the other hand, differ from discrete systems in that the mass and elasticity are continuously distributed. Such systems are also known as *distributed-parameter* systems, and examples include strings, rods, beams, plates and shells. While discrete systems possess a finite number of degrees of freedom, continuous systems have an infinite number of degrees of freedom because we need an infinite number of coordinates to specify the displacement of every point in an elastic body. The displacement in this case depends on two independent variables, namely x and t. As a result, the motion of continuous systems is governed by partial differential equations to be satisfied over the entire domain of the system, subject to boundary conditions and initial conditions.

Although discrete systems and continuous system may appear entirely different in nature, the difference is more in form than concept. As a matter of fact, a certain physical system can be modeled either as discrete or as distributed, depending on the objectives of the analysis. It turns out that discrete and continuous systems are indeed closely connected, and thus it comes as no surprise that both systems possess natural frequencies and normal modes of vibration.

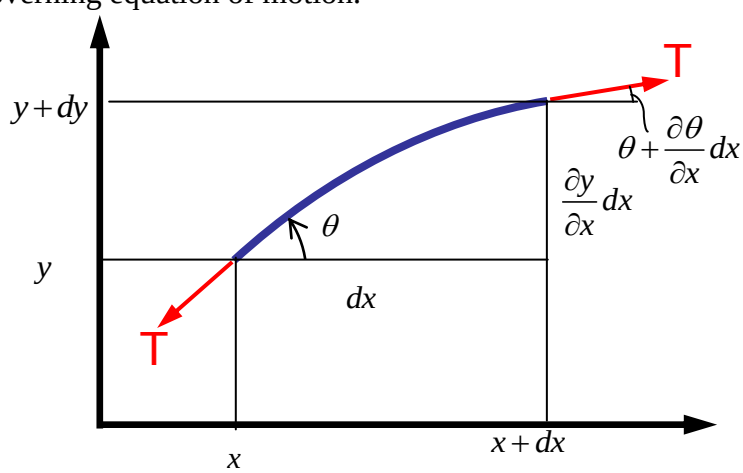
In this topic we will study the free and forced vibration of continuous systems. Emphasis will be placed on studying the vibration of taught strings, rods and beams. This covers a broad class of engineering applications, as many practical systems can be modeled by one or more of such elements in order to study the dynamic behavior

Vibration of Strings

The figure shows a fixed-fixed string of length L. The string is initially under tension T and the aim is to study the **transverse vibrations** denoted by the displacement $y(x,t)$, measured from the equilibrium position. It is assumed that both displacement and slope are small.



It is also assumed that the tension force in the string remains constant during vibration, which follows from the previous assumptions of small displacements. As in all continuous systems, the displacement variable depends on both the spatial (x) and temporal (t) coordinates. A free body diagram of a string element is shown below. Neglecting gravity effects, we can apply Newton's second law on the string element to obtain the governing equation of motion.



Applying $\uparrow \Sigma F_y = m \cdot a_y$ gives:

$$T \sin\left(\theta + \frac{\partial \theta}{\partial x} dx\right) - T \sin\theta = \rho dx \frac{\partial^2 y}{\partial t^2} \tag{1}$$

where ρ is the **mass per unit length** of the string. For small displacements, $\sin\theta \simeq \theta$, hence we obtain:

$$T \frac{\partial \theta}{\partial x} = \rho \frac{\partial^2 y}{\partial t^2} \tag{2}$$

But $\theta = \frac{\partial y}{\partial x}$ hence:

$$T\frac{\partial^2 y}{\partial x^2} = \rho\frac{\partial^2 y}{\partial t^2} \tag{3}$$

which can be written as:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \tag{4}$$

which is known as the one-dimensional wave equation and $c = \sqrt{\frac{T}{\rho}}$ is the velocity of wave propagation along the string. The wave equation is a partial differential equation, and the same form will be encountered in similar problems involving the dynamics of distributed-parameter models. The equation must be satisfied over the entire domain and is subject to **boundary conditions** as well as **initial conditions**. Accordingly, the problem posed is both a *boundary value problem (BVP)* and an *initial value problem (IVP)* from a mathematical point pf view.

We now seek the solution of the wave equation, which represents the variation of the transverse displacement at any point along the string and at any time for an arbitrary string that is set in motion by certain initial conditions and left to vibrate freely. This solution is emulated by the using the principle of separation of variables. In this way, the transverse displacement can be expressed as:

$$y(x,t) = Y(x)\cdot G(t) \tag{5}$$

It follows that:

$$\frac{\partial^2 y}{\partial x^2} = \frac{d^2 Y}{dx^2}\cdot G \tag{6}$$

and

$$\frac{\partial^2 y}{\partial t^2} = Y\cdot\frac{d^2 G}{dt^2} \tag{7}$$

Substitution into the equation of motion (4) yields:

$$\frac{d^2 Y}{dx^2}G = \frac{1}{c^2}Y\frac{d^2 G}{dt^2} \tag{8}$$

which can also be written as:

$$\frac{1}{Y}\frac{d^2 Y}{dx^2} = \frac{1}{c^2}\frac{1}{G}\frac{d^2 G}{dt^2} \tag{9}$$

It is noted that the left-hand-side (LHS) of the above equation depends **only** on the spatial variable x, whereas the RHS depends only on the temporal variable, t. In order to satisfy

the equation, both sides of equation (9) must be equal to a constant. Let this constant be $-(\omega/c)^2$. A negative constant was conveniently selected because this choice leads to an oscillatory motion. The choice of a zero or positive constant does not yield a vibratory motion, and therefore must be excluded. For example, if a zero constant was chosen, this leads to:

$$\frac{1}{c^2}\frac{1}{G}\frac{d^2G}{dt^2}=0$$

or

$$\frac{d^2G}{dt^2}=0$$

whose solution is given by:

$$G=c_1t+c_2$$

which is rejected because it indicates a solution that increases linearly with time. It can be shown that the choice of a positive constant gives rise to two terms; one exponentially increasing and the other exponentially decreasing.

Adopting the negative constant choice, and substituting into the equation of motion gives:

$$\frac{1}{Y}\frac{d^2Y}{dx^2}=-(\omega/c)^2 \tag{10}$$

which can be written as:

$$\frac{d^2Y}{dx^2}+(\omega/c)^2Y=0 \tag{11}$$

Furthermore,

$$\frac{1}{c^2}\frac{1}{G}\frac{d^2G}{dt^2}=-(\omega/c)^2 \tag{12}$$

which can similarly be expressed as:

$$\frac{d^2G}{dt^2}+\omega^2G=0 \tag{13}$$

These have the general solutions:

$$Y(x)=A\sin(\omega/c)x+B\cos(\omega/c)x \tag{14}$$

And

$$G(t)=C\sin\omega t+D\cos\omega t \tag{15}$$

The 4 constants A, B, C and D are to be determined from the boundary conditions (BC's) and initial conditions (IC's). It also worthy to note that equation (14) defines the deformation shape, whereas equation (15) defines the motion to be harmonic in time. It becomes appropriate now to define the unknown constant ω as the natural frequency of the system, and (ω/c) as the wave number or spatial frequency. The general solution may then be expressed as:

$$y(x,t) = \left(A\sin(\omega/c)x + B\cos(\omega/c)x\right)\cdot\left(C\sin\omega t + D\cos\omega t\right) \tag{16}$$

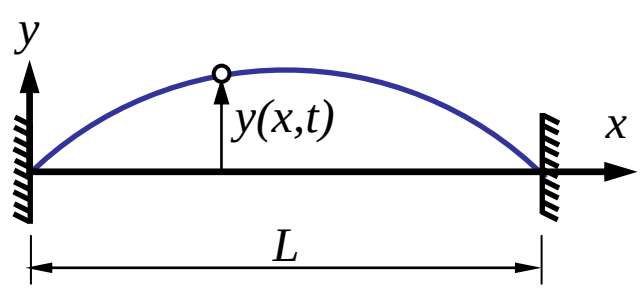
Alternatively, and after some algebraic manipulation, the above solution may also be written as:

$$\begin{aligned} y(x,t) = a_1\sin\left((\omega/c)x-\omega t\right) + a_2\cos\left((\omega/c)x-\omega t\right) + \\ a_3\sin\left((\omega/c)x+\omega t\right) + a_4\cos\left((\omega/c)x+\omega t\right) \end{aligned} \tag{17}$$

Once again, the solution must contain 4 unknown constants.

Example: Fixed-fixed string

Let us now consider the case of a string that is fixed at both ends, as shown.



The imposed boundary conditions indicate that the string displacement at both ends must be equal to zero, or:

$y(0,t) = 0$ and $y(L,t) = 0$. The general solution is:

$$\begin{aligned} y(x,t) &= Y(x)\cdot G(t) \\ &= \left(A\sin(\omega/c)x + B\cos(\omega/c)x\right)\cdot\left(C\sin\omega t + D\cos\omega t\right) \end{aligned}$$

Substitution of the first BC into the general solution gives:

$$0 = B\cdot\left(C\sin\omega t + D\cos\omega t\right)$$

which implies $B = 0$. The general solution hence becomes:

$$y(x,t) = \left(A\sin(\omega/c)x\right)\cdot\left(C\sin\omega t + D\cos\omega t\right)$$

Substitution of the second BC into the solution gives:

$$0 = \left(A\sin(\omega L/c)\right)\cdot\left(C\sin\omega t + D\cos\omega t\right)$$

which implies:

$$\sin(\omega L/c) = 0$$

hence

$$\omega L/c = n\pi \quad , \quad n = 1,2,3,\ldots$$

The above equation is termed the **frequency equation** or **characteristic equation** of the system, as it gives values of the system natural frequencies. Clearly, the system possesses an *infinite number of natural frequencies*, as suggested earlier. Having obtained the natural frequencies, the solution at any frequency or mode is expressed by:

$$\begin{aligned} y_n(x,t) &= \left(A_n\sin(n\pi x/L)\right)\left(C_n\sin\omega_n t + D_n\cos\omega_n t\right) \\ &= Y_n(x)\cdot G_n(t) \end{aligned}$$

Therefore, at each natural frequency, there corresponds a certain **mode shape** or an **eigenfunction** defined by

$$Y_n(x) = A_n\sin\left(n\pi x/L\right)$$

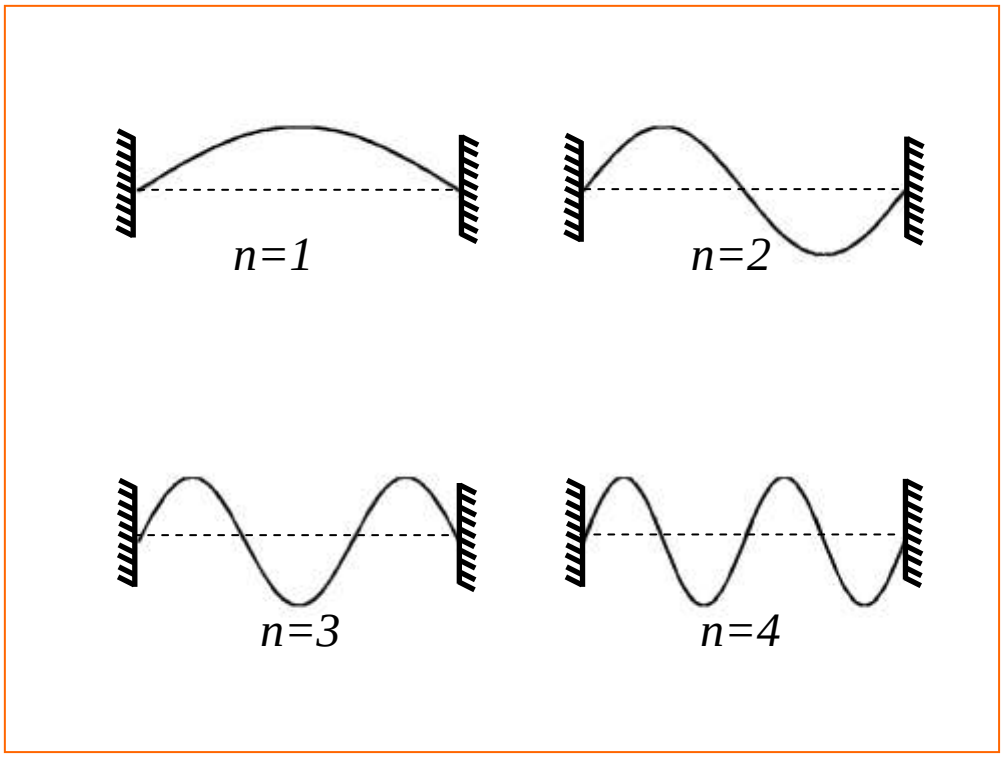
where each "*n*" represents a normal mode vibration with a natural frequency

$$\omega_n = \frac{n\pi c}{L}$$

and mode shape

$$Y_n(x) = A_n\sin\left(n\pi x/L\right)$$

where A_n are arbitrary constants. The figure below shows the first few modes of the string, as obtained from the above analysis.



The general solution is given by:

$$y_n(x,t) = \left(A_n \sin(n\pi x/L)\right)\left(C_n \sin\omega_n t + D_n \cos\omega_n t\right)$$

where the terms in the first bracket define the displacement pattern at mode "*n*", and the terms on the last bracket define a harmonic motion at the corresponding natural frequency. The free vibration solution is finally obtained as the sum of all modes of vibration, or:

$$y(x,t) = \sum_{n=1}^{\infty}\left(C_n \sin\omega_n t + D_n \cos\omega_n t\right)\sin(n\pi x/L)$$

where C_n, D_n are constants to be determined from the IC's.

The eigenfunctions can also be shown to possess an orthogonality property (see next section) which is given by:

$$\int_0^L Y_n(x)Y_m(x)dx = \begin{cases} 0 & n \neq m \\ h_n & n = m \end{cases}$$

For a fixed-fixed string, this becomes:

$$\int_0^L \sin(n\pi x/L)\sin(m\pi x/L)dx = \begin{cases} 0 & n \neq m \\ L/2 & n = m \end{cases}$$

In order to study the complete free vibration problem, the initial conditions must also be defined. Assume that the string is subjected to the following initial conditions:

$$y(x,0) = f(x) \quad , \quad \dot{y}(x,0) = g(x)$$

Substitution into the general solution gives:

$$f(x) = \sum_{n=1}^{\infty} D_n \sin(n\pi x/L)$$

and

$$g(x) = \sum_{n=1}^{\infty} C_n \omega_n \sin(n\pi x/L)$$

In order to obtain the constants C_n, D_n we multiply the above equations by $\sin(n\pi x/L)$ and integrate over the string length. By using the orthogonality codition, we get:

$$D_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)dx$$

and

$$C_n = \frac{2}{L\omega_n}\int_0^L g(x)\sin(n\pi x/L)dx$$

It thus follows that the constants C_n, D_n determine the **contribution** of each mode to the general solution. Now consider the special case where the initial conditions impose a displacement pattern that **coincides** with one of the natural modes, say mode "*k*", and that the initial velocity is zero. In this way, we have:

$$f(x) = \sin(k\pi x/L)$$

and

$$g(x) = 0$$

It follows that:

$$D_n = \begin{cases} 0 & k \neq n \\ 1 & k = n \end{cases}$$

and

$$C_n = 0$$

Therefore, the general solution reduces to:

$$y(x,t) = \sin(k\pi x/L)\cos\omega_k t$$

That is, the string vibrates in its "k^{th}" mode due to the fact that the initial conditions cause only the "k^{th}" mode to be excited. The continuous system in this special case behaves like a single DOF system. The same notion can be observed for a multi-DOF system. If, on the other hand, an impulse is given to the system, it can be shown that a wide spectrum of frequencies or modes are excited, and hence the system response will contain a summation of a large number of modes. In practice, the contribution of the higher modes is usually smaller, and the system response will be almost predominantly be described by the fundamental (i.e. first) mode, together with only a few higher modes.

Orthogonality of Eigenfunctions

Let us now prove the orthogonality condition for a fixed-fixed string. The same technique can be applied to other forms of boundary conditions. Recall Eq. (11) obtained previously:

$$\frac{d^2Y}{dx^2} + \left(\omega/c\right)^2 Y = 0 \tag{18}$$

This equation must be satisfied at all natural frequencies or all normal modes of vibration. Consequently, at an arbitrary mode "*m*", we have ω_m, Y_m as the natural frequency and associated eigenfuction, respectively. Similarly, at mode "*n*", we have ω_n, Y_n. Let us first assume that these two modes are distinct. It follows from Eq. (18) that:

$$-\frac{d^2Y_m}{dx^2} = \left(\omega_m/c\right)^2 Y_m = 0 \tag{19}$$

and

$$-\frac{d^2Y_n}{dx^2} = \left(\omega_n/c\right)^2 Y_n = 0 \tag{20}$$

Multiplying Eq. (19) by Y_n and integrating over the string length yields:

$$-\int_0^L Y_n \frac{d^2Y_m}{dx^2} dx = \int_0^L \left(\omega_m/c\right)^2 Y_m Y_n dx \tag{21}$$

The LHS of the above equation can be integrated by parts:

$$LHS = -\left[Y_n \frac{dY_m}{dx}\bigg|_0^L - \int_0^L \frac{dY_m}{dx}\frac{dY_n}{dx} dx \right] = \int_0^L \frac{dY_m}{dx}\frac{dY_n}{dx} dx \tag{22}$$

and the first term vanishes due to the boundary conditions defined (both ends fixed). It follows that:

$$\int_0^L \frac{dY_m}{dx}\frac{dY_n}{dx} dx = \int_0^L \left(\omega_m/c\right)^2 Y_m Y_n dx \tag{23}$$

Similarly, we can multiply equation (20) by Y_m and integrate over the string length. This yields:

$$\int_0^L \frac{dY_n}{dx}\frac{dY_m}{dx} dx = \int_0^L \left(\omega_n/c\right)^2 Y_m Y_n dx \tag{24}$$

It is noted that the LHS of Eqs. (23) and (24) are identical, irrespective of the order of multiplication. Subtracting Eq. (24) from Eq. (23) gives:

$$0 = \frac{\omega_m^2 - \omega_n^2}{c^2} \int_0^L Y_m Y_n dx \tag{25}$$

But ω_m, ω_n are two distinct modes, i.e. $\omega_m \neq \omega_n$, therefore:

$$\int_0^L Y_m Y_n dx = 0 \quad , \quad m \neq n \tag{26}$$

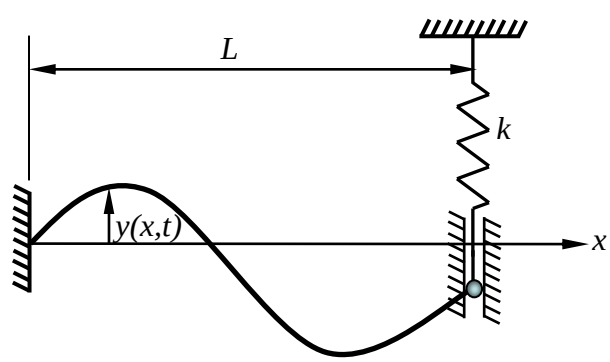
which proves the orthogonality condition for eigenfunctions of a fixed-fixed string. This condition also holds true for other types of BC's. Moreover, since eigenfunctions can be arbitrarily scaled (or *normalized*), we can write:

$$\int_0^L Y_m Y_m dx = h_m \quad , \quad m = n \tag{27}$$

where h_m is a constant.

Elastic or Inertial Attachments

Finally, let us now consider other forms of boundary conditions. Figure 6 shows a string that is fixed at one end and attached to a spring at the other. The boundary condition at the fixed end $x = 0$ is given by $y(0,t) = 0$. This is called a **geometric boundary condition**, because it describes a *specified* displacement. Such a condition is also known as an **essential** or **imposed** boundary condition.



Elastic attachments.

The boundary condition at the other end is not so obvious at first sight. Indeed it becomes appropriate to draw a free-body diagram of the string in order to investigate the force interaction. Such a free-body diagram is shown in Fig.7. At $x = L$ we need to balance forces in the vertical direction. Thus we have:

$$T\frac{\partial y}{\partial x}(L,t) = -ky(L,t) \tag{28}$$

This is called a **natural boundary condition** (also known as **dynamic** or **additional** boundary condition) as it describes forces and moments acting on the system. We can then proceed with the solution in the same way described above in order to obtain the natural frequencies, eigenfunctions and response to initial conditions.

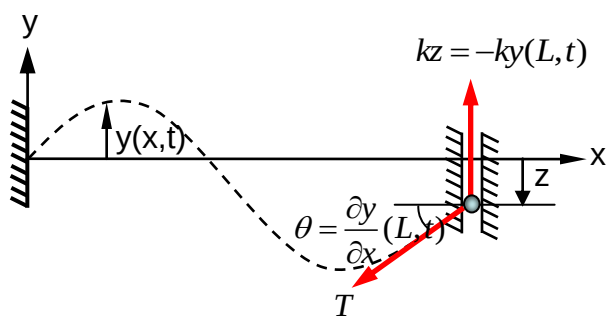
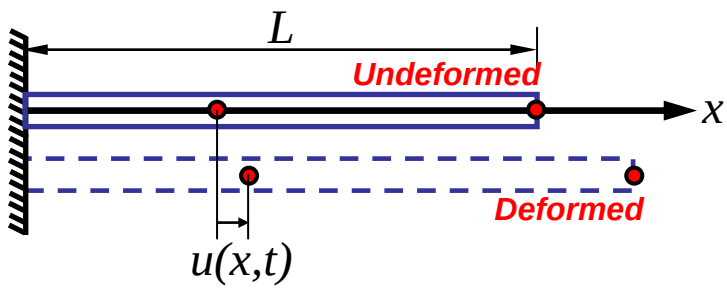


Figure 7. Free-body diagram.

Vibration of Rods

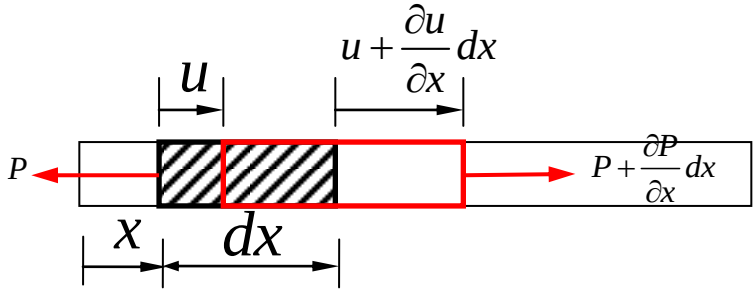
In this section, let us study the free **longitudinal vibration** of rods (bars). Consider a fixed-free rod of length L undergoing longitudinal vibration, as shown below.



The nomenclature adopted in this case is listed below.

ρ	Density (mass per unit volume)
P	Axial force
$u(x,t)$	Longitudinal displacement
A	Cross-sectional area
E	Young's modulus of elasticity

The longitudinal displacement, which is assumed to be small, depends on both the spatial (x) and temporal (t) variables. It is assumed that the displacement is small. In order to study the rod vibration, we need to write down its equation of motion. Consider a rod element, as shown in below.



An infinitesimal element of the rod, shown by the hatched area, undergoes longitudinal motion, and is drawn in its deformed configuration, as shown. The change in length of the element is expressed as:

$$\Delta L = \frac{\partial u}{\partial x} dx \tag{29}$$

The axial strain is then given by:

$$\varepsilon = \frac{\Delta L}{L} = \frac{\partial u}{\partial x} \tag{30}$$

The axial stress in the element can be written as:

$$\sigma = \frac{P}{A} \tag{31}$$

Applying Hooke's law, $\sigma = E\varepsilon$, yields:

$$\frac{P}{A} = E\frac{\partial u}{\partial x} \tag{32}$$

Rearranging:

$$EA\frac{\partial u}{\partial x} = P \tag{33}$$

Differentiating with respect to x gives:

$$EA\frac{\partial^2 u}{\partial x^2} = \frac{\partial P}{\partial x} \tag{34}$$

Now apply Newton's second law in the axial direction:

$$\frac{\partial P}{\partial x} dx = \left(\rho A dx\right)\frac{\partial^2 u}{\partial t^2} \tag{35}$$

Combining the previous equations gives:

$$EA\frac{\partial^2 u}{\partial x^2} = \rho A\frac{\partial^2 u}{\partial t^2} \tag{36}$$

which is rearranged to give:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{37}$$

where $c = \sqrt{\frac{E}{\rho}}$ is the velocity of wave propagation along the rod. Note the similarity between the wave equation of a string and that of a rod. Once again, the wave equation is a second order partial differential equation that must be satisfied over the entire rod domain, subject to boundary and initial conditions. Also note that the displacement is a function of two independent variables, *x* and *t*.

Solution of the wave equation is emulated by using separation of variables, and the process follows directly from that adopted for strings. Thus we seek a solution in the form:

$$u(x,t) = U(x) \cdot G(t) \tag{38}$$

Upon differentiating partially with respect to t and x yields:

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 U}{dx^2} \cdot G \tag{39}$$

and

$$\frac{\partial^2 u}{\partial t^2} = U \cdot \frac{d^2 G}{dt^2} \tag{40}$$

Substitution into the equation of motion gives:

$$\frac{d^2 U}{dx^2} G = \frac{1}{c^2} U \frac{d^2 G}{dt^2} \tag{41}$$

which is rearranged in the form:

$$\frac{1}{U} \frac{d^2 U}{dx^2} = \frac{1}{c^2} \frac{1}{G} \frac{d^2 G}{dt^2} \tag{42}$$

Once again, we note that the LHS depends only on *x*, whereas the RHS depends only on *t*. In order to satisfy this equation, both sides must be equal to a constant. Let this constant be $-(\omega/c)^2$ for oscillatory motion to prevail. It then follows that:

$$\frac{1}{U} \frac{d^2 U}{dx^2} = -(\omega/c)^2 \tag{43}$$

or:

$$\frac{d^2 U}{dx^2} + (\omega/c)^2 U = 0 \tag{44}$$

and

$$\frac{1}{c^2}\frac{1}{G}\frac{d^2G}{dt^2} = -\left(\omega/c\right)^2 \tag{45}$$

or:

$$\frac{d^2G}{dt^2} + \omega^2 G = 0 \tag{46}$$

These have the general solutions:

$$U(x) = A_1 \sin\left(\omega/c\right)x + A_2 \cos\left(\omega/c\right)x \tag{47}$$

and

$$G(t) = A_3 \sin \omega t + A_4 \cos \omega t \tag{48}$$

The solution $U(x)$ defines the deformation shape, whereas $G(t)$ defines the motion to be harmonic in time. The four constants A_1, A_2, A_3, A_4 are to be determined from the boundary and initial conditions. The natural frequency ω is yet to be determined, and the expression $\left(\omega/c\right)$ is known as the **wave number** or **spatial frequency**. The general solution is finally obtained by:

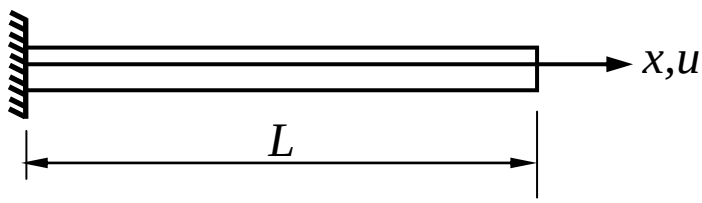
$$u(x,t) = \left(A_1 \sin(\omega/c)x + A_2 \cos(\omega/c)x\right)\cdot\left(A_3 \sin \omega t + A_4 \cos \omega t\right) \tag{49}$$

After some algebraic manipulation, the solution may also be expressed as:

$$\begin{aligned} y(x,t) = a_1 \sin\left((\omega/c)x - \omega t\right) + a_2 \cos\left((\omega/c)x - \omega t\right) + \\ a_3 \sin\left((\omega/c)x + \omega t\right) + a_4 \cos\left((\omega/c)x + \omega t\right) \end{aligned} \tag{50}$$

Example: Fixed-free Rod

As an example, let us investigate the case of a fixed-free rod.



The boundary conditions for this case are:

$$u(0,t)=0$$

which results in:

$$0=A_2\cdot G(t)$$

from which we get:

$$A_2=0$$

The general solution then becomes:

$$u(x,t)=\left(A_1\sin(\omega/c)x\right)\cdot G(t)$$

At the free end $x=L$ the axial force must vanish $P=0$. But

$$P=EA\frac{\partial u}{\partial x}=EA(\omega/c)\left(A_1\cos(\omega/c)x\right)\cdot G(t)$$

hence:

$$0=EA(\omega/c)\left(A_1\cos(\omega L/c)\right)\cdot G(t)$$

which implies:

$$\cos(\omega L/c)=0$$

which is the **frequency equation** or **characteristic equation** of the system. Solution of this equation is:

$$\omega L/c=\left(\frac{2n-1}{2}\right)\pi \quad , \quad n=1,2,3,\ldots$$

Hence the natural frequencies of the system are given by:

$$\omega_n=\left(\frac{2n-1}{2}\right)\frac{\pi c}{L} \quad , \quad n=1,2,3,\ldots$$

and the normal modes of vibration are:

$$U_n(x) = \left(A_{1n} \sin(\omega_n / c)x\right)$$

The solution of each mode becomes:

$$u_n(x,t) = A_{1n} \sin\left((2n-1)\pi x/2L\right) \cdot \left(C_{1n} \sin \omega_n t + D_{1n} \cos \omega_n t\right)$$

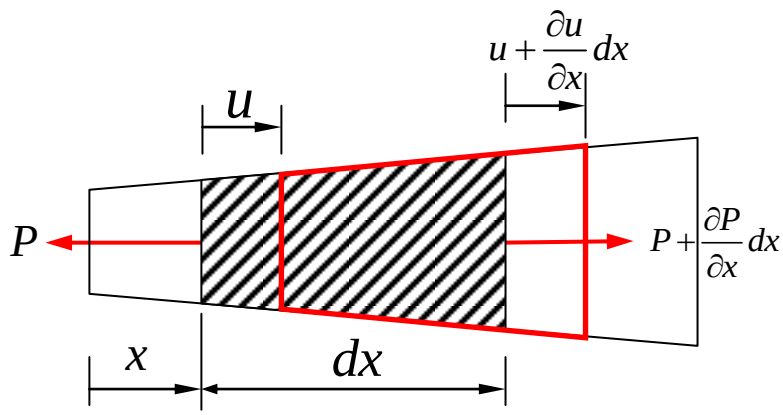
In other words, at each natural frequency, there corresponds a mode shape or an eigenfunction defined by:

$$U_n(x) = A_{1n} \sin\left((2n-1)\pi x/2L\right)$$

and each n represents a normal mode vibration with a natural frequency $\omega_n = \left(\frac{2n-1}{2}\right)\frac{\pi c}{L}$ where A_{1n} are arbitrary constants.

Rod with Non-uniform Cross Section

Consider a rod with a non-uniform cross section, as shown below. The equation of motion can be obtained using the same techniques described previously.



Applying Newton's second law yields:

$$\frac{\partial P}{\partial x} dx = \left(\rho A(x) dx\right) \frac{\partial^2 u}{\partial t^2} \tag{51}$$

which gives:

$$\frac{\partial P}{\partial x} = \left(\rho A(x)\right) \frac{\partial^2 u}{\partial t^2} \tag{52}$$

Note that $A(x)$ describes the variation of cross sectional area along the rod axis. Upon application of Hooke's law, we obtain:

$$P = EA(x) \frac{\partial u}{\partial x} \tag{53}$$

Differentiating with respect to x gives:

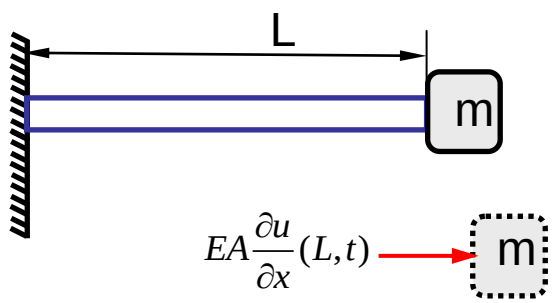
$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial x}\left(EA(x) \frac{\partial u}{\partial x} \right) \tag{54}$$

and hence the equation of motion is expressed as:

$$\rho A(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(EA(x)\frac{\partial u}{\partial x} \right) \tag{55}$$

Other Boundary Conditions

Finally, let us consider the case where inertial attachments are appended to the rod, as shown below.

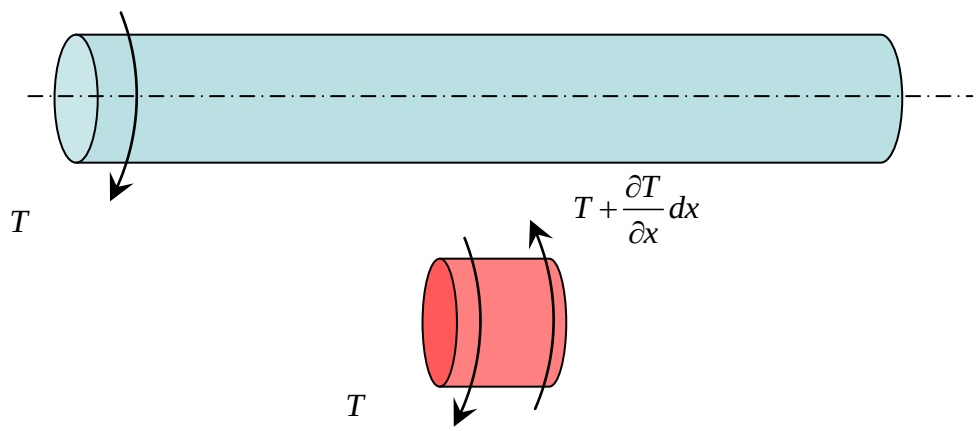


The boundary condition at the fixed end is straight forward. Examination of the boundary condition at the free end leads us to write the equation of motion of the attached mass using Newton's second law. The resulting equation can be expressed as:

$$EA\frac{\partial u}{\partial x}(L,t) = -m\frac{\partial^2 u}{\partial t^2}(L,t) \tag{56}$$

Torsional Vibration of Rods

Now consider a rod that is subjected to a twisting moment T, as shown below.



The angle of twist can be expressed as:

$$d\theta = \frac{Tdx}{GJ} \tag{57}$$

where

θ : angle of twist

G : modulus of rigidity

J : polar moment of inertia

We can then write:

$$\frac{\partial T}{\partial x}dx = GJ\frac{\partial^2 \theta}{\partial x^2}dx \tag{58}$$

Applying Newton's law to the rod element, we obtain:

$$dxGJ\frac{\partial^2 \theta}{\partial x^2} = \rho J dx\frac{\partial^2 \theta}{\partial t^2} \tag{59}$$

hence

$$\frac{\partial^2 \theta}{\partial t^2} = \frac{G}{J}\frac{\partial^2 \theta}{\partial x^2} \tag{60}$$

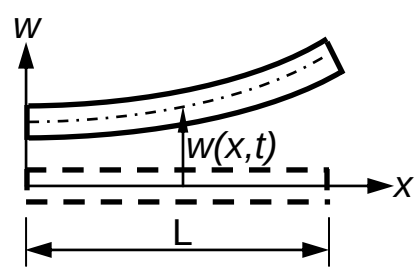
which is in the form:

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \theta}{\partial t^2} \tag{61}$$

with $c = \sqrt{\frac{G}{\rho}}$ representing the wave velocity. It is noted that this equation has exactly the same form as equation (37) representing the axial vibration of rods, with the $u \to \theta$, $E \to G$. Thus the evaluating of the natural frequencies, mode shapes and system response follows directly from the equations previously mentioned.

Vibration of Beams

This section deals with the **transverse vibration** of beams. The figure shows an elastic beam drawn in both the undeformed and deformed configurations. Although the figure suggests a cantilever arrangement, the analysis is suited for arbitrary boundary conditions. Transverse displacements, measured from the neutral axis at equilibrium, are designated as *w(x,t)*.



From strength of materials, and adopting the Euler-Bernoulli beam theory, we have:

$$M = -EI\frac{\partial^2 w}{\partial x^2} \tag{62}$$

and:

$$V = \frac{\partial M}{\partial x} \tag{63}$$

where

$E =$ Young's modulus of elasticity [N/m^2]

$I =$ Second moment of area [m^4]

$M =$ bending moment [Nm]

$w =$ Transverse displacement [m]

$V =$ Shear force [N]

Consider an infinitesimal beam element as shown in Fig. 15.

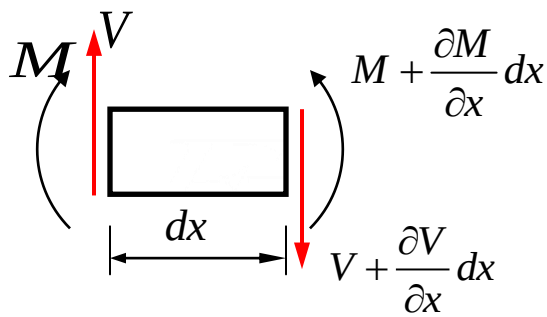


Figure 15. Forces and moments acting on a beam element.

Neglecting rotary inertia, we can apply Newton's law in the vertical (transverse) direction to obtain the equation of motion:

$$\frac{\partial V}{\partial x} dx = \rho A dx \frac{\partial^2 w}{\partial t^2} \tag{64}$$

where A is the cross-sectional area. From equations (57) and (58), we have:

$$V = \frac{\partial M}{\partial x} = \frac{\partial}{\partial x}\left(-EI \frac{\partial^2 w}{\partial x^2} \right) \tag{65}$$

For constant EI we obtain:

$$V = -EI \frac{\partial^3 w}{\partial x^3} \tag{66}$$

or:

$$\frac{\partial V}{\partial x} = -EI \frac{\partial^4 w}{\partial x^4} \tag{67}$$

Combining equation (59) with (62) yields the equation that governs the free transverse vibration of a uniform elastic beam as:

$$-EI\frac{\partial^4 w(x,t)}{\partial x^4} = \rho A\frac{\partial^2 w(x,t)}{\partial t^2} \tag{68}$$

Solution of the above equation can be emulated using the technique of separation of variables, as described in previous sections. In this way, the solution can be expressed as:

$$w(x,t) = W(x)F(t) \tag{69}$$

Substituting (64) into (63) yields:

$$-\frac{EI}{\rho A}\frac{1}{W}\frac{d^4W}{dx^4} = \frac{1}{F}\frac{d^2F}{dt^2} \tag{70}$$

We observe that the left side of (65) depends *only* on x, while the right side depends *only* on t. Because x and t are independent variables, we conclude that both sides of (65) must be equal to a constant. Let this constant be $-\omega^2$. It follows that:

$$\frac{d^2F}{dt^2} + \omega^2 F = 0 \tag{71}$$

which has a solution in the form:

$$F(t) = C_1 \sin\omega t + C_2 \cos\omega t \tag{72}$$

where C_1 and C_2 are constants to be determined from the initial conditions.

Furthermore, we have:

$$\frac{d^4W}{dx^4} - \frac{\rho A}{EI}\omega^2 W = 0 \tag{73}$$

Denoting $\beta^4 = \frac{\rho A}{EI}\omega^2$ we get:

$$\frac{d^4W}{dx^4} - \beta^4 W = 0 \tag{74}$$

The solution of (69) can be shown to be:

$$W(x) = A_1 \sin\beta x + A_2 \cos\beta x + A_3 \sinh\beta x + A_4 \cosh\beta x \tag{75}$$

where A_1, A_2, A_3 and A_4 are constants to be determined from the boundary conditions.

1.12 Example: Cantilever Beam

The boundary conditions for a cantilever beam are given by:

At $x = 0, \quad W = 0, \quad \dfrac{dW}{dx} = 0$

At $x = L, \quad \dfrac{d^2W}{dx^2} = 0, \quad \dfrac{d^3W}{dx^3} = 0$

Upon substitution into (70), and after some algebraic manipulation, we obtain the *eigenfunctions for a cantilever beam* as:

$$W_n(x) = A_n\left[\sin\beta_n x - \sinh\beta_n x - \frac{\sin\beta_n L + \sinh\beta_n L}{\cos\beta_n L + \cosh\beta_n L}\left(\cos\beta_n x - \cosh\beta_n x\right)\right] \tag{76}$$

where β_n is obtained by solving

$$\cos\beta L \cosh\beta L = -1 \tag{77}$$

Equation (72) can be solved numerically to give the *eigenvalues* $\beta_1 L, \beta_2 L, \ldots, \beta_n L$. The first three solutions can be shown to be:

$$\beta_1 = 1.8751/L$$
$$\beta_2 = 4.6941/L$$
$$\beta_3 = 7.8548/L$$

Now the natural frequencies can be obtained from $\omega_n = \beta_n^2\sqrt{\dfrac{EI}{\rho A}}$ and the first three values are obtained as:

$$\omega_1 = \left(1.8751\right)^2\sqrt{\frac{EI}{\rho AL^4}}$$
$$\omega_2 = \left(4.6941\right)^2\sqrt{\frac{EI}{\rho AL^4}}$$
$$\omega_3 = \left(7.8548\right)^2\sqrt{\frac{EI}{\rho AL^4}}$$

Figure 16 shows a typical frequency response plot of a cantilever beam, acted upon by a harmonic force at its tip. The amplitude of the tip motion is plotted as a function of the excitation frequency. The peaks occur at the *natural frequencies* of the system, and the deformation pattern of the beam (eigenfunctions) at each frequency is plotted at the top.

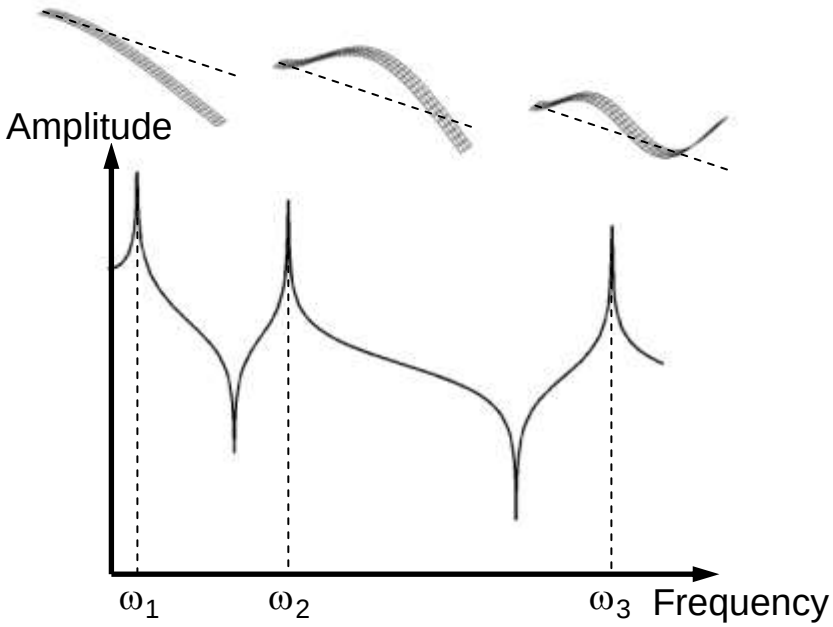
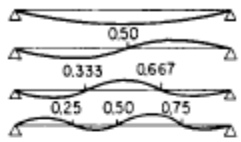
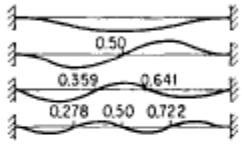
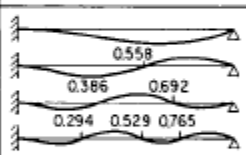
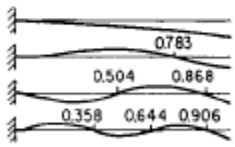
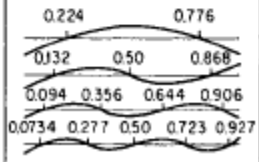


Figure 16. Frequency response of a beam.

Table 1 shows gives a summary of the boundary conditions, natural frequencies and normal modes of commonly-encountered uniform beam configurations, as taken from *Harris' Shock and Vibration Handbook, 5th edition, 2002*.

Table 1. Properties of uniform beams.

SUPPORTS	MODE n	(A) SHAPE AND NODES (NUMBERS GIVE LOCATION OF NODES IN FRACTION OF LENGTH FROM LEFT END)	(B) BOUNDARY CONDITIONS EQ (7.16)	(C) FREQUENCY EQUATION	(D) CONSTANTS EQ (7.16)	(E) kl EQ (7.14) $\omega_n = k^2\sqrt{\frac{EIg}{A\gamma}}$	(F) R RATIO OF NON-ZERO CONSTANTS COLUMN (D)
HINGED-HINGED	1		$x=0\begin{cases}X=0\\X''=0\end{cases}$	SIN kl=0	A = 0	3.1416	1.0000
	2				B = 0	6.283	1.0000
	3		$x=l\begin{cases}X=0\\X''=0\end{cases}$		$\frac{C}{D}=1$	9.425	1.0000
	4					12.566	1.0000
	n>4					$\approx n\pi$	1.0000
CLAMPED-CLAMPED	1		$x=0\begin{cases}X=0\\X'=0\end{cases}$	(COS kl)(COSH kl) = 1	A = 0	4.730	-0.9825
	2				C = 0	7.853	-1.0008
	3		$x=l\begin{cases}X=0\\X'=0\end{cases}$		$\frac{D}{B}=R$	10.996	-1.0000-
	4					14.137	-1.0000+
	n>4					$\approx\frac{(2n+1)\pi}{2}$	-1.0000-
CLAMPED-HINGED	1		$x=0\begin{cases}X=0\\X'=0\end{cases}$	TAN kl = TANH kl	A = 0	3.927	-1.0008
	2				C = 0	7.069	-1.0000+
	3		$x=l\begin{cases}X=0\\X''=0\end{cases}$		$\frac{D}{B}=R$	10.210	-1.0000
	4					13.352	-1.0000
	n>4					$\approx\frac{(4n+1)\pi}{4}$	-1.0000
CLAMPED-FREE	1		$x=0\begin{cases}X=0\\X'=0\end{cases}$	(COS kl)(COSH kl) = -1	A = 0	1.875	-0.7341
	2				C = 0	4.694	-1.0185
	3		$x=l\begin{cases}X''=0\\X'''=0\end{cases}$		$\frac{D}{B}=R$	7.855	-0.9992
	4					10.996	-1.0000+
	n>4					$\approx\frac{(2n-1)\pi}{2}$	-1.0000-
FREE-FREE	1		$x=0\begin{cases}X''=0\\X'''=0\end{cases}$	(COS kl)(COSH kl) = 1	B = 0	0 (REPRESENTS TRANSLATION)	
	2				D = 0	4.730	-0.9825
	3		$x=l\begin{cases}X''=0\\X'''=0\end{cases}$		$\frac{C}{A}=R$	7.853	-1.0008
	4					10.996	-1.0000-
	5					14.137	-1.0000+
	n>5					$\approx\frac{(2n-1)\pi}{2}$	-1.0000-

1.13 Forced Response

The solution of the forced vibration problem for beams, as well as other continuous systems, can be emulated through the principle of mode superposition. The equation of motion for forced vibration is:

$$EI\frac{\partial^4 w(x,t)}{\partial x^4}+\rho A\frac{\partial^2 w(x,t)}{\partial t^2}=f(x,t) \tag{78}$$

The solution is assumed to be in the form:

$$w(x,t)=\sum_{n=1}^{\infty}W_n(x)q_n(t) \tag{79}$$

where $W_n(x)$ is the nth normal mode or eigenfunction of the beam satisfying the differential equation (73) which is:

$$\frac{d^4W_n(x)}{dx^4}-\frac{\rho A}{EI}\omega_n{}^2W_n(x)=0 \tag{80}$$

and $q_n(t)$ is the generalized coordinate for the nth mode. By substituting (79) into (78) we get:

$$EI\sum_{n=1}^{\infty}\frac{d^4W_n(x)}{dx^4}q_n(t)+\rho A\sum_{n=1}^{\infty}W_n(x)\frac{d^2q_n(t)}{dt^2}=f(x,t) \tag{81}$$

Dividing by ρA yields:

$$\frac{EI}{\rho A}\sum_{n=1}^{\infty}\frac{d^4W_n(x)}{dx^4}q_n(t)+\sum_{n=1}^{\infty}W_n(x)\frac{d^2q_n(t)}{dt^2}=\frac{f(x,t)}{\rho A} \tag{82}$$

But from (80) we have:

$$\frac{d^4W_n(x)}{dx^4}=\frac{\rho A}{EI}\omega_n{}^2W_n(x) \tag{83}$$

Inserting (83) into (82) yields:

$$\sum_{n=1}^{\infty}\omega_n{}^2W_n(x)q_n(t)+\sum_{n=1}^{\infty}W_n(x)\frac{d^2q_n(t)}{dt^2}=\frac{f(x,t)}{\rho A} \tag{84}$$

Multiplying (84) throughout by $W_m(x)$ and integrating along the beam length and using the orthogonality conditions, we obtain:

$$\frac{d^2 q_n(t)}{dt^2} + \omega_n{}^2 q_n(t) = \frac{1}{\rho A b} Q_n(t) \tag{85}$$

where $Q_n(t)$ is the generalized force corresponding to $q_n(t)$ and is given by:

$$Q_n(t) = \int_0^l f(x,t) W_n(x) dx \tag{86}$$

and the constant b is given by:

$$b = \int_0^l W_n^2(x) dx \tag{87}$$

The solution of (85) can be obtained using the Duhamel integral as:

$$q_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t + \frac{1}{\rho A b \omega_n} \int_0^t Q_n(\tau) \sin \omega_n (t-\tau) d\tau \tag{88}$$

The first 2 terms on the right hand side of (88) represent the transient or free vibration solution (resulting from initial conditions) and the 3rd term denotes the steady-state vibration (resulting from the forcing function).

1.14 Example

Find the steady-state response of a simply-supported beam that is subjected to a harmonic force $f(x,t) = F_o \sin \omega t$ applied at $x = a$ as shown in Fig. 17.

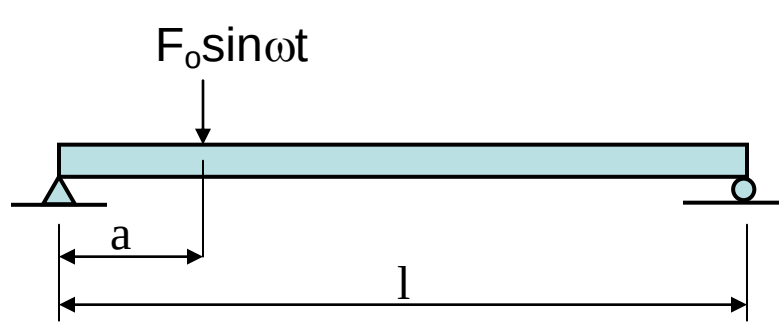


Figure 17. Simply-supported beam under harmonic excitation.

The normal modes of a simply-supported beam are expressed as:

$$W_n(x) = \sin\frac{n\pi x}{l}$$

Consequently, the generalized force is obtained as:

$$Q_n(t) = \int_0^l F_o \sin\omega t \sin\frac{n\pi x}{l}\,dx = F_o \sin\frac{n\pi a}{l}\sin\omega t$$

The steady-state response of the beam is then obtained from (88) as:

$$q_n(t) = \frac{1}{\rho A b \omega_n}\int_0^t Q_n(\tau)\sin\omega_n(t-\tau)\,d\tau$$

with

$$b = \int_0^l W_n^2(x)\,dx = \int_0^l \sin^2\frac{n\pi x}{l}\,dx = \frac{l}{2}$$

The solution can be expressed as:

$$q_n(t) = \frac{2F_o}{\rho A l}\frac{\sin\frac{n\pi a}{l}}{\omega_n^2 - \omega^2}\sin\omega t$$

Hence the response is obtained from (79) as:

$$w(x,t) = \frac{2F_o}{\rho A l}\sum_{n=1}^{\infty}\frac{1}{\omega_n^2 - \omega^2}\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l}\sin\omega t$$