Finite Element Vibration Analysis

Introduction

In previous topics we learned how to model the dynamic behavior of multi-DOF systems, as well as systems possessing infinite numbers of DOF. As the reader may realize, our discussion was limited to rather simple geometries and boundary conditions, mainly because more complicated geometries tend to have progressively more complicated exact solutions, or even no exact solutions at all. The vast majority of geometries which occur in practical engineering applications are complicated three-dimensional continua which cannot be represented adequately by the simple closed-form mathematical models. These theoretical models also cannot portray the appreciable non-linear or anisotropic material characteristics which are often met with in practice. In such situations therefore, we must resort to numerical methods in which the continuum of infinitesimal material particles is represented by an approximately equivalent assembly of inter-connected discrete elements which are each so simple that they can be treated individually as mathematical continua. There are a number of methods whereby such models can be analyzed such as numerical solution of differential equations, finite differences, finite elements, boundary elements, relaxation techniques, and so on. In this topic, we will demonstrate the Finite Element Method (FEM) as a typical powerful approach which can handle vibration analysis.

In essence, the FE technique is a numerical technique in which a continuous elastic structure, or continuum, is divided (discretized) into small but finite substructures, known as elements. Elements are interconnected at nodes. In this way, a continuum with infinite number of degrees of freedom can be modeled with a set of elements having a finite number of degrees of freedom. It is noted that while each finite element represents a continuous system by itself possessing infinite number of DOF, we can choose the size of the element to be small enough, so that the deformation within the finite element can be approximated (interpolated) by relatively low-order polynomials.

In this topic, we will discuss the application of FE procedures in vibration analysis, with emphasis on developing equations of motion for elementary geometries, developing FE codes on MATLAB and applying commercial FE software to handle more advanced geometries.

Finite Element Analysis of Rods

Consider an elastic uniform rod of total length L_R as shown in Fig.1. Upon applying the FE technique, the rod is discretzed into a finite number of elements. As the rod under investigation is uniform, it is assumed that all elements used to mesh the complete rod are identical. In more advanced problems, of course, this technique may not be possible, as the entity to be meshed often has a more complicated contour or geometry, which makes it difficult to employ identical elements. The technique described herein, however, can be applied to non-identical elements.

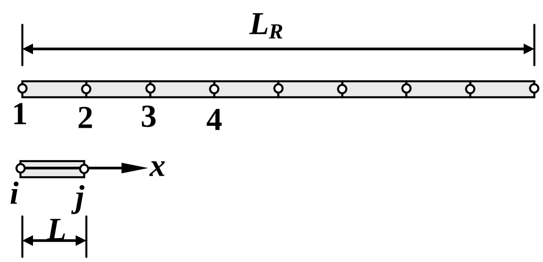

Figure 1. Uniform rod.

Let $u(x)$ denote the axial displacement within an element. It is recognized that displacements are also functions of *time,* but for the sake of simplicity in writing the equations, the time t is dropped, but the reader must remember that time dependence is indeed retained in the upcoming analysis. Note also that the rod element by itself is treated as a continuous structural member, meaning that axial displacement within the element is a continuous function. An essence of the finite element technique is that field variables, in this case axial displacements, within each element is interpolated (or approximated) in terms of the nodal quantities, that is the displacements measured at the two nodes bounding the element. In this way, a whole structure is regarded as a multi-

DOF system, with field variables evaluated only the nodes. Once these quantities are known, the displacement field within the entire structure can be evaluated using the interpolation functions initially assumed for the structure. Several interpolation functions can be chosen to properly satisfy the physics of the problem.

For the problem at hand, the displacement field within the rod element is approximated by a linear function:

$$u(x) = \alpha_1 + \alpha_2 x \quad , \quad 0 \le x \le L \tag{1}$$

This can be written as:

$$u(x) = \{1 \quad x\} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} \tag{2}$$

In this way, the **vector of nodal degrees of freedom** is expressed as:

$$\{\delta^e\} = \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \tag{3}$$

But we have:

$$\begin{aligned} u_i &= \alpha_1 \\ u_j &= \alpha_1 + \alpha_2 L \end{aligned} \tag{4}$$

Therefore:

$$\{\delta^e\} = \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \left[\bar{A}\right] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} \tag{5}$$

from which:

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \left[\bar{A}\right]^{-1} \{\delta^e\} \tag{6}$$

Performing the matrix inverse yields:

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/L & 1/L \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \tag{7}$$

Substituting Eq. (7) into (2) gives:

$$u(x) = \{1 \quad x\} \begin{bmatrix} 1 & 0 \\ -1/L & 1/L \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \tag{8}$$

or:

$$u(x) = \left\{1 - \frac{x}{L} \quad \frac{x}{L}\right\} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \tag{9}$$

which is written as:

$$u(x) = \{N(x)\}\{\delta^e\} \tag{10}$$

The vector $\{N(x)\}$ is called the vector of **interpolation functions** or **shape functions**, and is typical in all finite element procedures. Once again, using this vector, we can express the displacement at any point within the rod element in terms of the nodal degrees of freedom.

Several methods can be employed to obtain the equation of motion of rod elements. In this section we will use Lagrange's equation. In this regard, we need to express the strain energy associated with the rod element. The reader is referred to any elementary textbook in strength of materials to express the strain energy as:

$$U = \frac{1}{2} EA \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx \tag{11}$$

Inserting the shape functions, as determined in (10) and noting that:

$$\left(\frac{\partial u}{\partial x}\right)^2 = \{\delta^e\}^T \{N_x\}^T \{N_x\}\{\delta^e\}$$

we obtain:

$$U = \frac{1}{2}\{\delta^e\}^T EA \int_0^L \{N_x\}^T \{N_x\} dx \{\delta^e\} \tag{12}$$

where

$$\{N_x\} = \left\{\frac{\partial N(x)}{\partial x}\right\}$$

The strain energy can then be written as:

$$U = \frac{1}{2}\{\delta^e\}^T \left[K^e\right]\{\delta^e\} \tag{13}$$

where $\left[K^e\right]_{2\times2}$ is the **element stiffness matrix**, given by:

$$\left[K^e\right] = EA\int_0^L \{N_x\}^T \{N_x\} dx \tag{14}$$

Performing the above calculation yields:

$$\left[K^e\right] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{15}$$

which is the element stiffness matrix for a uniform rod element.

The element kinetic energy is then evaluated for the rod element and can be expressed as:

$$T = \frac{1}{2}\rho A\int_0^L \dot{u}^2 dx \tag{16}$$

Inserting the shape functions, as determined in (10) yields:

$$T = \frac{1}{2}\left\{\dot{\delta}^e\right\}^T \rho A\int_0^L \{N\}^T \{N\} dx \left\{\dot{\delta}^e\right\} \tag{17}$$

which can also be written as:

$$T = \frac{1}{2}\left\{\dot{\delta}^e\right\}^T \left[M^e\right]\left\{\dot{\delta}^e\right\} \tag{18}$$

where $\left[M^e\right]_{2\times2}$ is the **element mass matrix**, given by:

$$\left[M^e\right] = \rho A\int_0^L \{N\}^T \{N\} dx \tag{19}$$

Performing the above calculation yields:

$$\left[M^e\right] = \rho AL\begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \tag{15}$$

which is the element mass matrix for a uniform rod element. Now we can apply Lagrange's equation, which is in the form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} = 0 \tag{16}$$

or more precisely:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\left\{\dot{\delta}^e\right\}}\right)-\frac{\partial T}{\partial\left\{\delta^e\right\}}+\frac{\partial V}{\partial\left\{\delta^e\right\}}=0 \qquad (17)$$

The resulting equations of motion are therefore:

$$\left[M^e\right]\left\{\ddot{\delta}^e\right\}+\left[K^e\right]\left\{\delta^e\right\}=0 \qquad (18)$$

which is the element equation of motion for free vibration. Upon **assembly** of the element equations of motion (see next section), we can determine the equations of motion for the entire structure in the form:

$$\left[M\right]\left\{\ddot{\delta}\right\}+\left[K\right]\left\{\delta\right\}=0 \qquad (19)$$

We can easily extend the derivation to include a damping term and a forcing term in the equation of motion:

$$\left[M\right]\left\{\ddot{\delta}\right\}+\left[C\right]\left\{\dot{\delta}\right\}+\left[K\right]\left\{\delta\right\}=f(t) \qquad (20)$$

Assembly of Equations of Motion

Assembly of the element equations of motion can be accomplished by the routine assembly process used in finite element procedures. As an example, consider a rod consisting of 2 elements as shown in Fig.2.

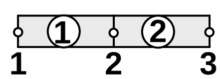

Figure 2. Rod with 2 elements.

Under static conditions, we have for element 1:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}=\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad (21)$$

whereas for element 2:

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}=\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \qquad (22)$$

For the whole rod, we have:

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{23}$$

where the principle of superposition has been applied to obtain the stiffness matrix for the entire structure. The same procedure is used to assemble the mass matrix, and for any number of elements. In this way, the stiffness and mass matrices are essentially banded matrices, with the properties of each element inserted as we proceed into the assembly process. Elements having different material properties or geometrical parameters (cross sectional areas) can easily be assembled.

Boundary Conditions

Inspection of the overall stiffness matrix indicated in the previous section reveals that it is essentially *singular*, meaning that upon application of an external force one cannot solve for the unknown displacements (in static analysis, for instance) using $[K]\{\delta\} = \{F\}$ by multiplying by the inverse of the stiffness matrix, since it is non-existent. That is so because the structure in this case is not restrained, and hence application of an external force causes instability in the analysis. Such an error is often encountered in commercial FE software, where the user is alerted of a "singular stiffness matrix" error, which is an indication that the structure has insufficient constraints.

Application of boundary conditions can be accomplished using the same procedures typically adopted in static FE analysis. As an illustrative example, consider the rod shown in Fig.3, where one end is rigidly fixed to a wall, while the other is subjected to a force.

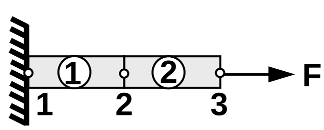

Figure 3. Rod with one end fixed , other end loaded.

In this case, we have:

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{24}$$

However, F_1 is an unknown reaction force that is yet to be determined. Although we can directly obtain the value of this reaction from physical intuition (equal and opposite to the applied tip load), we cannot make this intervention before using the FE analysis to solve the problem! Moreover, in more complicated cases, such as statically indeterminate problems, we cannot obtain the reaction forces using only the equilibrium equations. As far as displacements are concerned, we will also regard u_2 and u_3 as unknown displacements. Hence equation (24) represents three equations in three unknowns. In order to proceed with the solution, we will first eliminate the equations pertaining to unknown reaction forces, in this case the first equation. This is accomplished by striking the first element in both the load vector and displacement vector, as well as striking the first row and column of the stiffness matrix. To this end, we will get :

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \tag{25}$$

which can be solved for the unknown displacements $\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$ as the vector of externally applied loads $\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$ is known. Once the displacements are known, we can then substitute into equation (24) with:

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} \tag{26}$$

in order to obtain the unknown reaction force. The procedure outlined in this elementary example can be generalized for all problems discussed in this topic. It is worthy to mention that the same is applied for the mass matrix, and all calculations are done on the reduced matrices.

Vibration Analysis of Rods

Let us now study the free and forced vibration analysis of rods using the finite element procedure outline in the previous sections. For free vibration, we have:

$$[M]\{\ddot{\delta}\}+[K]\{\delta\}=0 \tag{27}$$

Imposing harmonic motion in the form:

$$\{\delta\}=\{\Delta\}\sin\omega t \tag{28}$$

we get:

$$[K]\{\Delta\}=[M]\omega^2\{\Delta\} \tag{29}$$

hence:

$$[M]^{-1}[K]\{\Delta\}=\omega^2\{\Delta\} \tag{30}$$

which is clearly an eigenvalue problem that can be solved to obtain the natural frequencies and mode shapes of the rod under investigation. It is perhaps worthy to mention at this point that such an analysis can be readily made for any boundary conditions imposed on the structure, which is in clear contrast with exact analytical solutions that largely depend on the boundary conditions.

For steady-state harmonic analysis, on the other hand, we have:

$$[M]\{\ddot{\delta}\}+[K]\{\delta\}=\{f(t)\} \tag{31}$$

with:

$$\{f(t)\}=\{F\}\sin\omega t \tag{32}$$

The resulting displacement, described by equation (28) is inserted into (32) yielding:

$$\left[[K]-\omega^2[M]\right]\{\Delta\}=\{F\} \tag{33}$$

which can be solved for the steady-state displacement amplitude at each excitation frequency:

$$\{\Delta\}=\left[[K]-\omega^2[M]\right]^{-1}\{F\} \tag{34}$$

where the matrix on the right hand side is commonly known as the **dynamic stiffness matrix.**

As an illustrative example, consider a fixed-free rod with the following properties:

Modulus of elasticity	80 GPa
Cross-sectional area	0.01 m^2
Length	8 m
Density	7800 kg/m^3

Table 1 lists the first three natural frequencies as we increase the number of elements used, together with the analytical solution. Convergence is seen to take place as we increase the number of elements used for meshing the rod.

Table 1. Natural frequencies and comparison with analytical solution.

Number of elements	ω_1 [Hz]	ω_2 [Hz]	ω_3 [Hz]
1	110.4	---	---
2	102.7	358.7	---
3	101.2	331.1	600.6
4	100.7	317.7	577.2
5	100.5	311.4	551.8
6	100.4	308	536.4
10	100.2	303	513.3
20	100.1	300.9	503.6
40	100.1	300.4	501.2
Analytical Solution	100.0801	300.2	500.4

Figure 4 shows the first three mode shapes, as predicted by the present FE analysis.

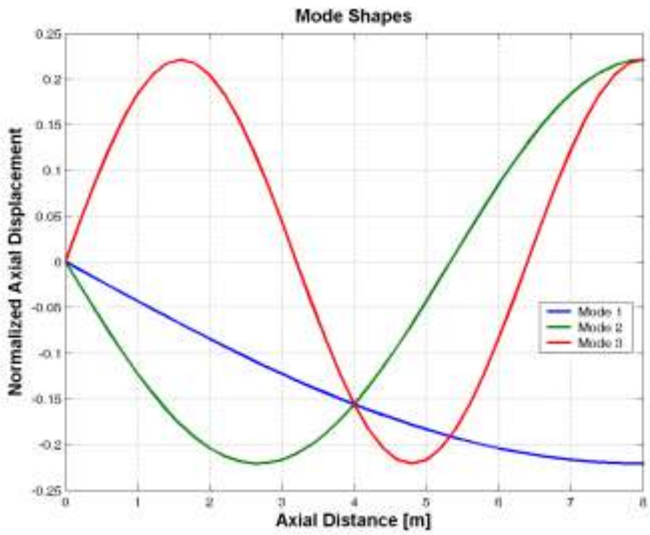

Figure 4. First three mode shapes of a fixed-free rod.

Consider now the case when the rod is subjected to a harmonic force acting at its tip and having a magnitude of 100N. We can determine the steady-state displacement amplitude at any point on the rod in accordance with the technique outlined previously. Figure 5 shows such frequency response, where the resonance is shown to occur at the natural frequencies indicated.

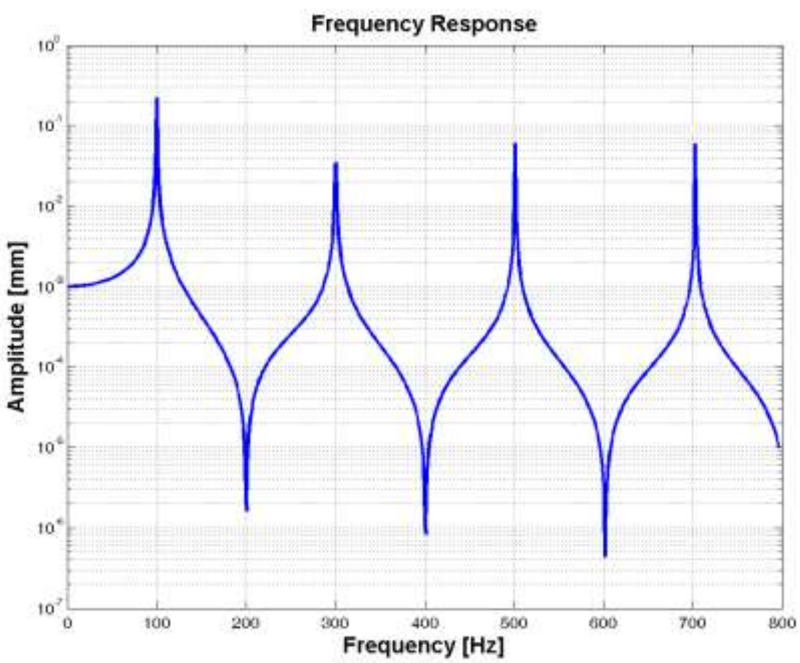

Figure 5. Frequency response of harmonically excited rod.

Damping can easily be implemented in the present model. Figure 6 shows the frequency response with damping included in the form of Rayleigh (proportional) damping.

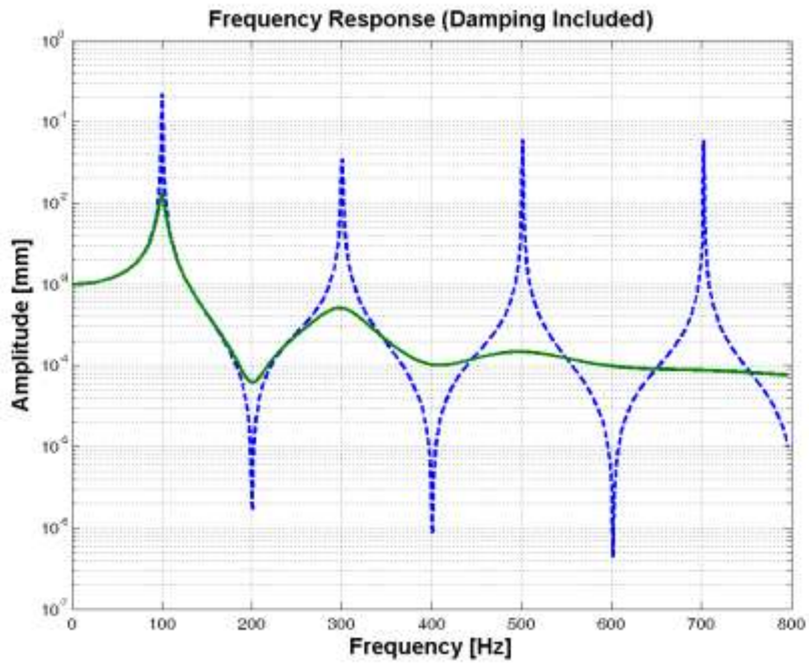

Figure 6. Frequency response of harmonically excited rod with damping.

5.6 Finite Element Analysis of Beams

The finite element analysis of beams follows the same procedure as that described for rods, except for the energy expressions, shape functions and element degrees of freedom. Figure 7 shows a beam in flexural vibrations, where the transverse displacement is expressed as $w(x)$. Once again, displacements are in fact functions of time, but the time t is dropped for brevity in the following analysis.

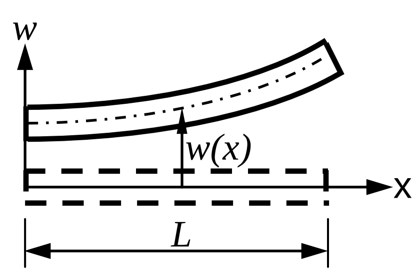

Figure 7. Beam in flexural vibrations.

For beams, the displacement field within the rod element is approximated by a cubic polynomial:

$$w(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 \quad , \quad 0 \le x \le L \qquad (35)$$

The choice of this function is based upon relations from strength of materials governing the Euler-Bernoulli beam theory, in which an element loaded by concentrated forces at its nodes features a linear variation in bending moment. Since bending moment is a function of the second derivative of transverse displacement, the above polynomial is justified.

Equation (35) can be written as:

$$w(x) = \left\{1 \quad x \quad x^2 \quad x^3\right\} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \qquad (36)$$

The **vector of nodal degrees of freedom** is then expressed as:

$$\left\{\delta^e\right\} = \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix} \qquad (37)$$

where

$$\theta = \frac{\partial w}{\partial x} \qquad (38)$$

is the slope. Substituting to get the values of the element degrees of freedom at each node, we have:

$$\begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix} = \left[\bar{A}\right] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \qquad (39)$$

from which:

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \left[\bar{A}\right]^{-1} \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix} \tag{40}$$

Substituting (40) into (36) gives:

$$w(x) = \begin{Bmatrix} 1 & x & x^2 & x^3 \end{Bmatrix} \left[\bar{A}\right]^{-1} \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix} \tag{41}$$

which is written as:

$$w(x) = \{N(x)\}\{\delta^e\} \tag{42}$$

Once again, the vector $\{N(x)\}$ is the vector of interpolation functions or shape functions for beam elements, which is used to express the displacement at any point within the rod element in terms of the nodal degrees of freedom.

The strain energy of the beam element is expressed as:

$$U = \frac{1}{2} EI \int_0^L \left(\frac{\partial^2 w}{\partial x^2} \right)^2 dx \tag{43}$$

Inserting the shape functions, as determined in (42) yields:

$$U = \frac{1}{2} \{\delta^e\}^T EI \int_0^L \{N_{xx}\}^T \{N_{xx}\} dx \{\delta^e\} \tag{44}$$

where

$$\{N_{xx}\} = \left\{ \frac{\partial^2 N(x)}{\partial x^2} \right\}$$

The strain energy of the beam element can be written as:

$$U = \frac{1}{2} \{\delta^e\}^T \left[K^e\right] \{\delta^e\} \tag{45}$$

where $\left[K^e\right]_{4\times 4}$ is the **element stiffness matrix**, given by:

$$\left[K^e\right] = EI \int_0^L \{N_{xx}\}^T \{N_{xx}\} dx \tag{46}$$

Performing the above calculation yields:

$$\left[K^e\right]=\frac{EI}{L^2}\begin{bmatrix}12/L & 6 & -12/L & 6\\ & 4L & -6 & 2L\\ & & 12/L & -6\\ & & & 4L\end{bmatrix} \qquad (47)$$

which is the element stiffness matrix for a uniform beam element.

The element kinetic energy of a beam element is expressed as:

$$T=\frac{1}{2}\rho A\int_0^L \dot{w}^2 dx \qquad (48)$$

Inserting the shape functions, as determined in (42) yields:

$$T=\frac{1}{2}\left\{\dot{\delta}^e\right\}^T \rho A\int_0^L \left\{N\right\}^T\left\{N\right\}dx\left\{\dot{\delta}^e\right\} \qquad (49)$$

which can also be written as:

$$T=\frac{1}{2}\left\{\dot{\delta}^e\right\}^T\left[M^e\right]\left\{\dot{\delta}^e\right\} \qquad (50)$$

where $\left[M^e\right]_{4\times4}$ is the **element mass matrix**, given by:

$$\left[M^e\right]=\rho A\int_0^L \left\{N\right\}^T\left\{N\right\}dx \qquad (51)$$

Performing the above calculation yields:

$$\left[M^e\right]=\rho AL\begin{bmatrix}13/35 & 11L/210 & 9/70 & -13L/420\\ & L^2/105 & 13L/420 & -L^2/140\\ & & 13/35 & -11L/210\\ & & & L^2/105\end{bmatrix} \qquad (52)$$

which is the element mass matrix for a uniform beam element. Applying Lagrange's equation yields the element equation of motion fro free vibration as:

$$\left[M^e\right]\left\{\ddot{\delta}^e\right\}+\left[K^e\right]\left\{\delta^e\right\}=0 \qquad (53)$$

Upon assembly of the element equations of, we can determine the equations of motion for the entire structure in the form:

$$[M]\{\ddot{\delta}\}+[K]\{\delta\}=0 \tag{54}$$

Or more generally:

$$[M]\{\ddot{\delta}\}+[C]\{\dot{\delta}\}+[K]\{\delta\}=f(t) \tag{55}$$

5.7 Examlple

Consider a cantilever beam with the following properties:

Modulus of elasticity	80 GPa
Cross-section (width x height)	20 mm × 0.4 mm
Length	0.2 m
Density	2700 kg/m^3

Table 2 lists the first three natural frequencies as we increase the number of elements used, together with the analytical solution. Once again, convergence is seen to take place as we increase the number of elements used for meshing the rod.

Table 2. Natural frequencies and comparison with analytical solution.

Number of elements	ω_1 [Hz]	ω_2 [Hz]	ω_3 [Hz]
2	8.7974	55.5732	187.9587
3	8.7940	55.2866	156.2198
4	8.7934	55.1698	155.4919
5	8.7932	55.1331	154.8515
6	8.7932	55.1192	154.5796
10	8.7931	55.1074	154.3365
Analytical Solution	8.7922	55.1034	154.3461

Figure 8 shows the first three mode shapes, as predicted by the present FE analysis.

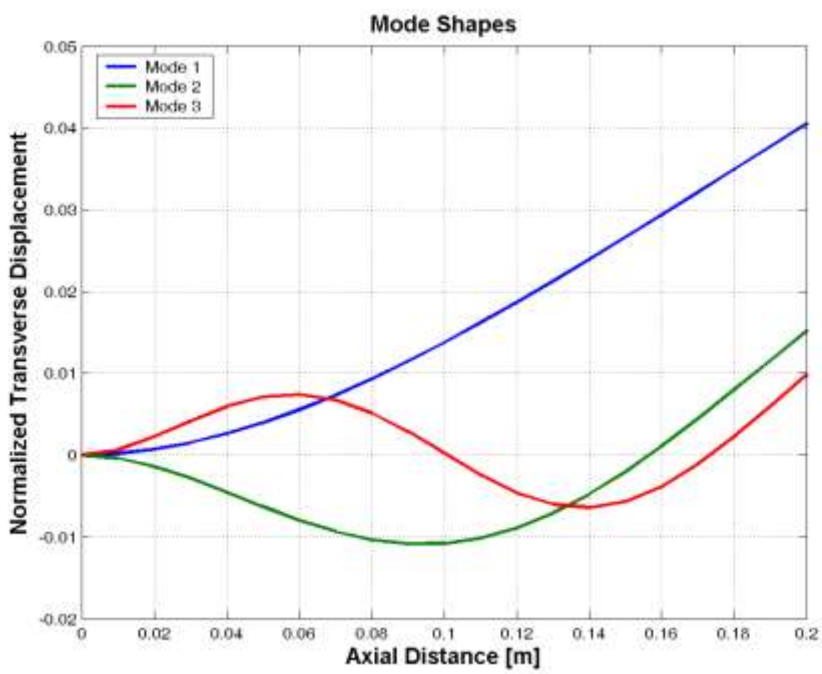

Figure 8. Cantilever beam mode shapes.