Response of a Damped System under Harmonic Force

The equation of motion is written in the form:

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t \qquad (1)$$

Note that F_0 is the amplitude of the driving force and ω is the driving (or forcing) frequency, not to be confused with ω_n. Equation (1) is a non-homogeneous, 2nd order differential equation. This will have two solutions: the homogeneous (F_0=0) and the particular (the periodic force), with the total response being the sum of the two responses. The homogeneous solution is the free vibration problem from last chapter. We will assume that the particular solution is of the form:

$$x_p(t) = A_1 \sin \omega t + A_2 \cos \omega t \qquad (2)$$

Thus the particular solution is a *steady-state* oscillation having the same frequency ω as the exciting force and a phase angle, as suggested by the sine and cosine terms. Taking the derivatives and substituting into (1) we get:

$$\left(k - m\omega^2\right)\left(A_1 \sin \omega t + A_2 \cos \omega t\right) + c\omega\left(A_1 \cos \omega t - A_2 \sin \omega t\right) = F_0 \cos \omega t$$

Equating the coefficients of the sine and the cosine terms, we get two equations:

$$c\omega A_1 + \left(k - m\omega^2\right) A_2 = F_0$$

$$\left(k - m\omega^2\right) A_1 - c\omega A_2 = 0$$

This leads to a solution of:

$$A_1 = \frac{F_0 c\omega}{\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2}, \quad A_2 = \frac{F_0\left(k - m\omega^2\right)}{\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2}$$

Aside: The equation

$$x_p(t) = A_1 \sin \omega t + A_2 \cos \omega t$$

can also be written as:

$$x_p(t) = X \cos(\omega t - \phi)$$

To convert between the 2 forms, i.e. to get the constants X and ϕ, substitute $t=0$ and equate the displacements and velocities in both equations. This yields:

$$A_2 = X \cos\phi \quad , \quad A_1 = X \sin\phi$$

Thus

$$X = \sqrt{A_1^2 + A_2^2} \quad , \quad \tan\phi = \frac{A_1}{A_2}$$

The solution can then be written in the form:

$$x_p(t) = X \cos(\omega t - \phi)$$

where

$$X = \frac{F_0}{\sqrt{\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2}} \, , \phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right)$$

using ω_n and ζ from before and introducing X_0 and r to be

$$\omega_n = \sqrt{\frac{k}{m}}, \zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{mk}}, \frac{c}{m} = 2\zeta\omega_n$$

$$X_0 = \frac{F_0}{k} = \text{deflection under static force } \mathrm{F}_0$$

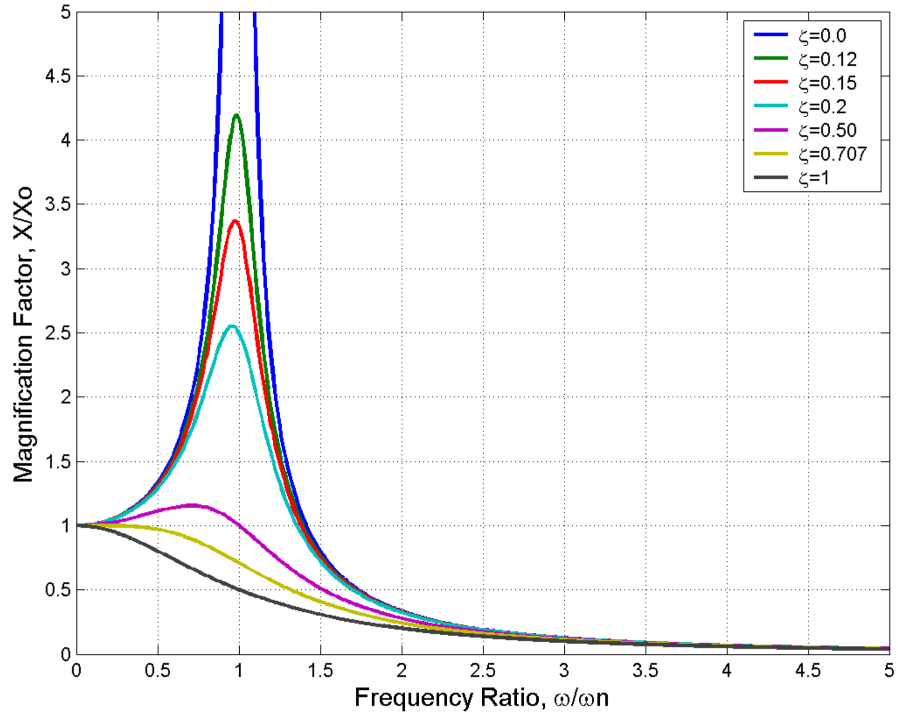
$$r = \frac{\omega}{\omega_n} = \text{frequency ratio}$$

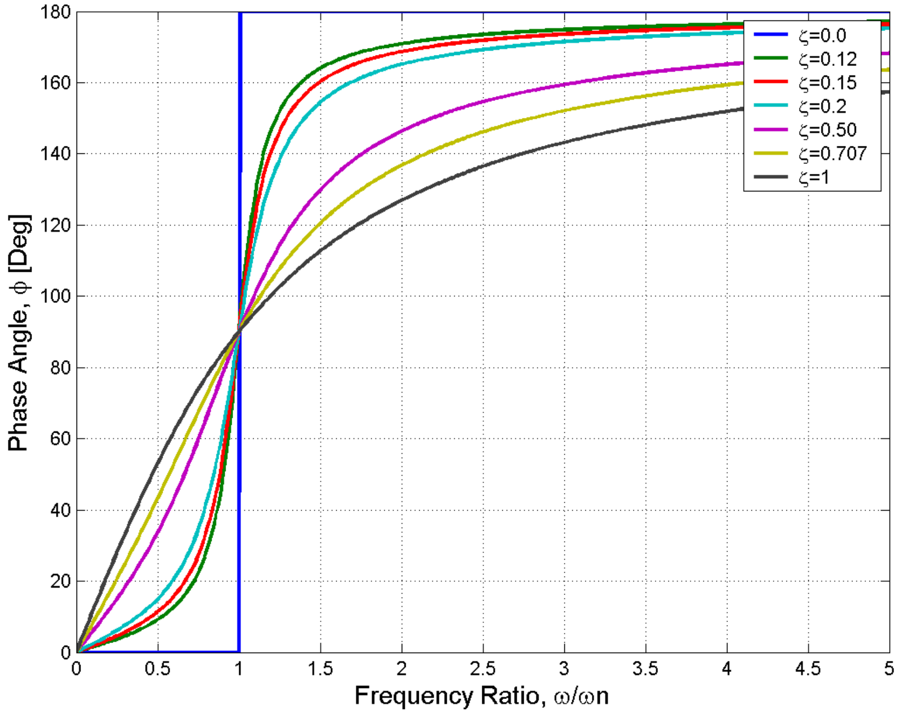
we can write

$$M = \frac{X}{X_o} = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}} = \frac{1}{\sqrt{\left(1 - r^2\right)^2 + \left(2\zeta r\right)^2}}$$

where M is the magnification factor (amplitude ratio) and

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$





See figure 3.11

The total response of the system is the sum of the homogeneous solution plus the particular solution or:

$$x(t) = x_h(t) + x_p(t) = X_0 e^{-\zeta\omega_n t}\cos(\omega_d t - \phi_o) + X\cos(\omega t - \phi)$$

Note that the homogeneous solution $x_h(t)$ *dies out with time*, and the steady-state solution prevails as long as the forcing function is present.

To solve this, you find $x_p(0)$ and $v_p(0)$ and then find X_0 and ϕ_0 such that the true initial conditions match $x(0)$ and $v(0)$.

Complex Analysis

Using the fact that the complex exponential is periodic, we can look at the real component of

$$m\ddot{x} + c\dot{x} + kx = F_0 e^{i\omega t}$$

we can assume a solution of the form:

$$x_p(t) = Xe^{i(\omega t - \phi)}$$

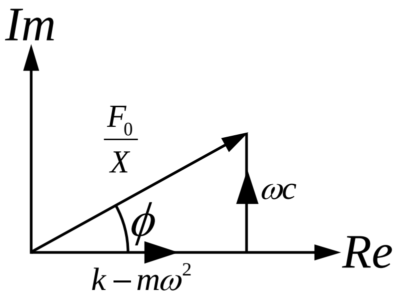
This leads to the equation:

$$\left(-m\omega^2 + c\omega i + k\right) Xe^{i(\omega t - \phi)} = F_0 e^{i\omega t}$$

which can be written as:

$$\left[\left(k - m\omega^2\right) + i\omega c\right] = \frac{F_0}{X} e^{i\phi}$$

This can be represented vectorially as:



from which we can get:

$$\frac{F_0}{X} = \sqrt{\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2}$$

so

$$X = \frac{F_0}{\sqrt{\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2}}$$

and

$$\phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right)$$

which is the same solution we had for the cosine case before, with a lot less work.

Base Motion

The equation of motion of the system is:

$$m\ddot{x}+c(\dot{x}-\dot{y})+k\left(x-y\right)=0$$

I. Relative Motion

Sometimes we are concerned with the relative motion of the mass with respect to the base. (Example: accelerometer and the velocity meter). In this case, we can define $z=x-y$. Our equation of motion becomes:

$$m\ddot{x}+c(\dot{x}-\dot{y})+k\left(x-y\right)=0$$

If the periodic input is in the form

$$y=Y\sin\omega t$$

the equation of motion becomes:

$$m\ddot{z}+c\dot{z}+kz=-m\ddot{y}=\underbrace{m\omega^2 Y}_{P_0}\sin\omega t$$

which is identical to the previous case (harmonic force), with F_0 replaced by P_0. The steady state solution can be written as:

$$z(t)=Z\sin\left(\omega t-\phi\right)$$

where

$$\frac{Z}{Z_0}=\frac{1}{\sqrt{\left(1-r^2\right)^2+\left(2\zeta r\right)^2}}$$

But

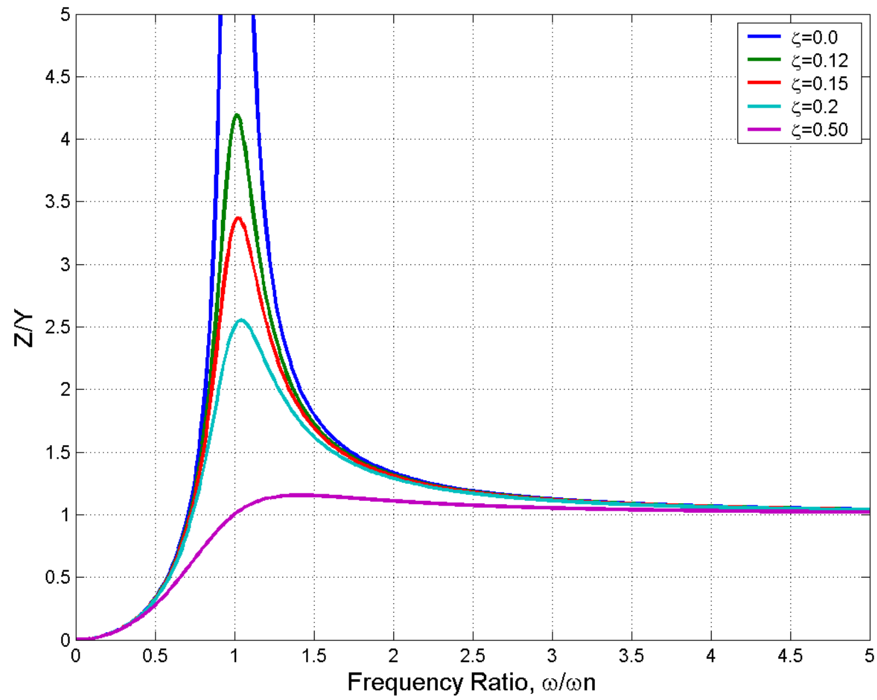
$$Z_0=\frac{P_0}{k}=\frac{m\omega^2 Y}{k}=Yr^2$$

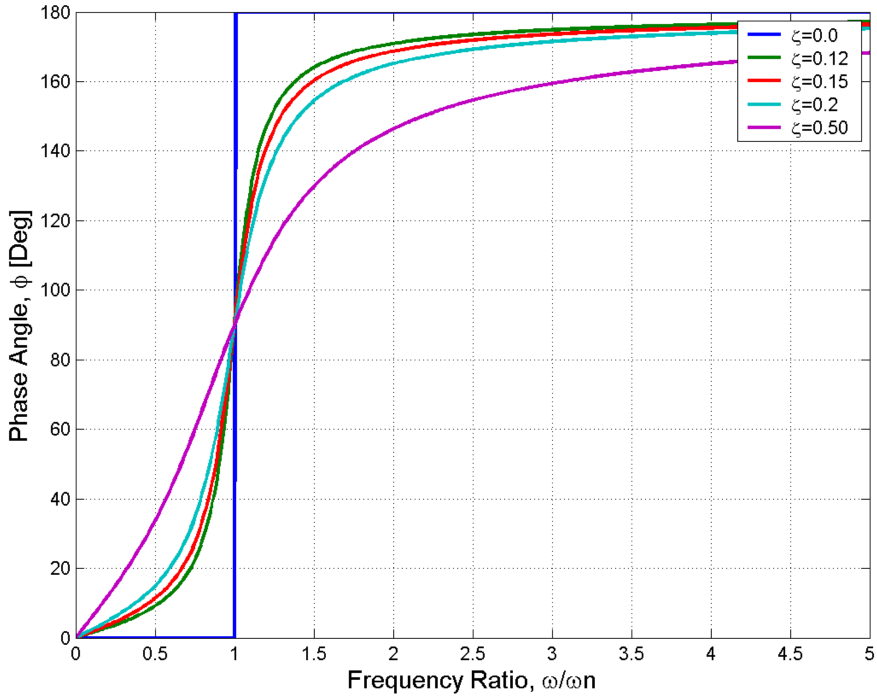
hence the solution is:

$$\boxed{\frac{Z}{Y}=\frac{r^2}{\sqrt{\left(1-r^2\right)^2+\left(2\zeta r\right)^2}}}$$

and

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$





Application: vibration measuring instruments

For negligible damping ($\zeta \ll 1$), we have

$$\frac{Z}{Y} = \frac{r^2}{1-r^2}$$

Case 1: $r \ll 1 \quad \Rightarrow \quad Z = Yr^2 = Y\frac{\omega^2}{\omega_n^2}$

At "low" frequencies, Z is proportional to $\omega^2 Y$, i.e. proportional to acceleration. In this range the accelerometer works. It must have a high ω_n such that ω/ω_n is small .

Case 2: $r \gg 1 \quad \Rightarrow \quad Z = Y$

At "high" frequencies, the ratio between *Z* and *Y* is one, i.e. Z is equal to the displacement. In this range the vibrometer works. It must have a low ω_n such that ω/ω_n is high.

I. Absolute Motion

If we are concerned with the absolute motion, we can write the equation of motion as:

$$m\ddot{x} + c\dot{x} + kx = ky + c\dot{y}$$

Here it becomes more convenient to assume the base motion having the form*:

$$y = Ye^{i\omega t}$$

We seek a solution in the form:

$$x = Xe^{i(\omega t - \phi)}$$

Substituting into the equation of motion yields:

$$\left(-m\omega^2 + c\omega i + k\right) Xe^{i(\omega t-\phi)} = \left[k + i\omega c\right] Ye^{i\omega t}$$

from which

$$\frac{X}{Y} e^{-i\phi} = \frac{k + i\omega c}{\left(k - m\omega^2\right) + i\omega c}$$

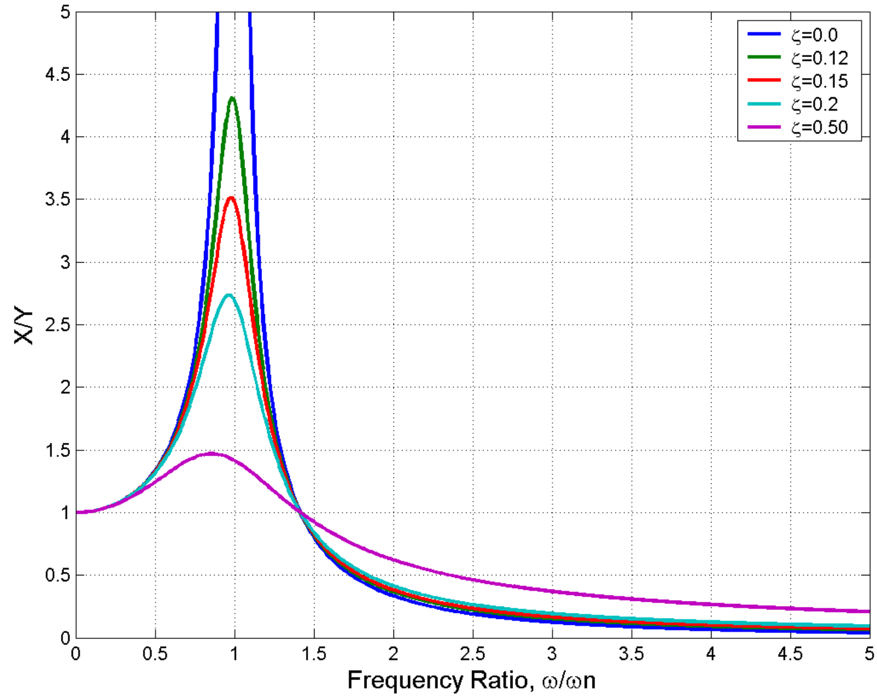
hence

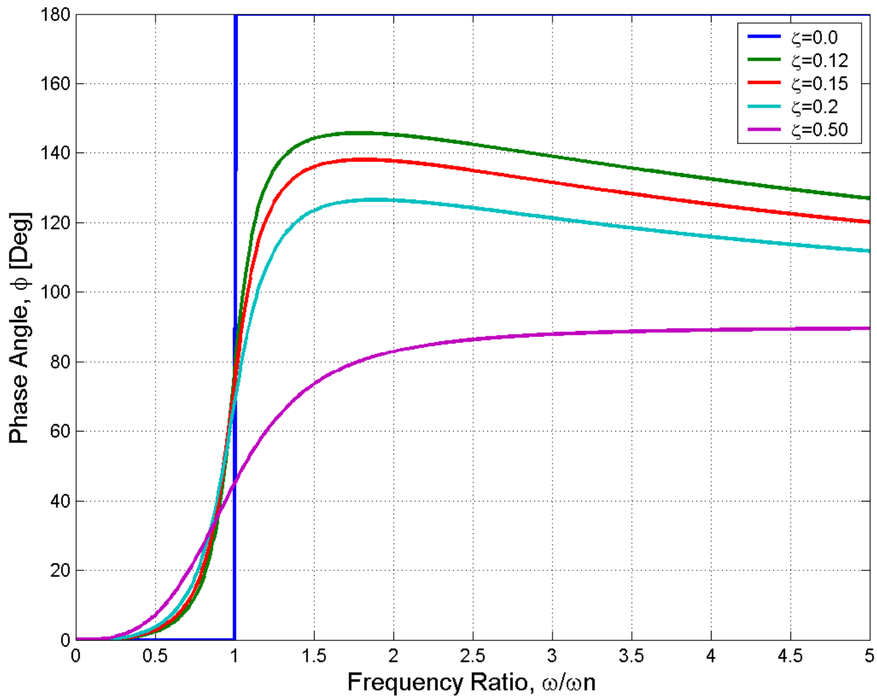
$$\left|\frac{X}{Y}\right| = \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{\left(1 - r^2\right)^2 + (2\zeta r)^2}}$$

and

$$\phi = \tan^{-1}\left(\frac{2\zeta r^3}{1 - r^2 + (2\zeta r)^2}\right)$$

* We can also assume a periodic base input of the form $y = Y \sin wt$, see page 11.





Note: the motion transmitted is less than 1 for $r > \sqrt{2}$. Hence for vibration isolation, we must have $\omega/\omega_n > \sqrt{2}$, i.e. ω_n must be small compared to ω.

The above solution can also be obtained if we assume a periodic input of

$$y = Y\sin\omega t$$

the equation of motion can be written as

$$m\ddot{x} + c\dot{x} + kx = kY\sin\omega t + c\omega Y\cos\omega t$$

$$m\ddot{x} + c\dot{x} + kx = A\sin(\omega t - \alpha)$$

where

$$A = Y\sqrt{k^2 + (c\omega)^2}, \alpha = \tan^{-1}\left(\frac{c\omega}{k}\right) \text{ (see page 2)}$$

But the above equation is in the same format now as the forced vibration one, with an amplitude of A instead of F_0 and an additional phase α. So we can find (see page 2)

$$x = X\sin(\omega t - \alpha - \phi_1) = \frac{Y\sqrt{k^2 + (c\omega)^2}}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}\sin(\omega t - \alpha - \phi_1)$$

$$\phi_1 = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right)$$

We can write the ratio of the amplitudes as

$$\frac{X}{Y} = \sqrt{\frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2}} = \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

using the same definition for ζ and *r* as before, we can also write the total solution as

$$x_p(t) = X\sin(\omega t - \phi)$$

where

$$\phi = \tan^{-1}\left(\frac{mc\omega^3}{k(k - m\omega^2) + (\omega c)^2}\right) = \tan^{-1}\left(\frac{2\zeta r^3}{1 + (4\zeta^2 - 1)r^2}\right)$$

which is the same solution obtained using complex analysis.

Force Transmitted

To find the force on the base during base motion, we have:

$$F = k(x-y)+c(\dot{x}-\dot{y}) = -m\ddot{x}$$

but x is known from before, so F is given by:

$$F = m\omega^2 X\sin(\omega t-\phi) = F_T\sin(\omega t-\phi)$$

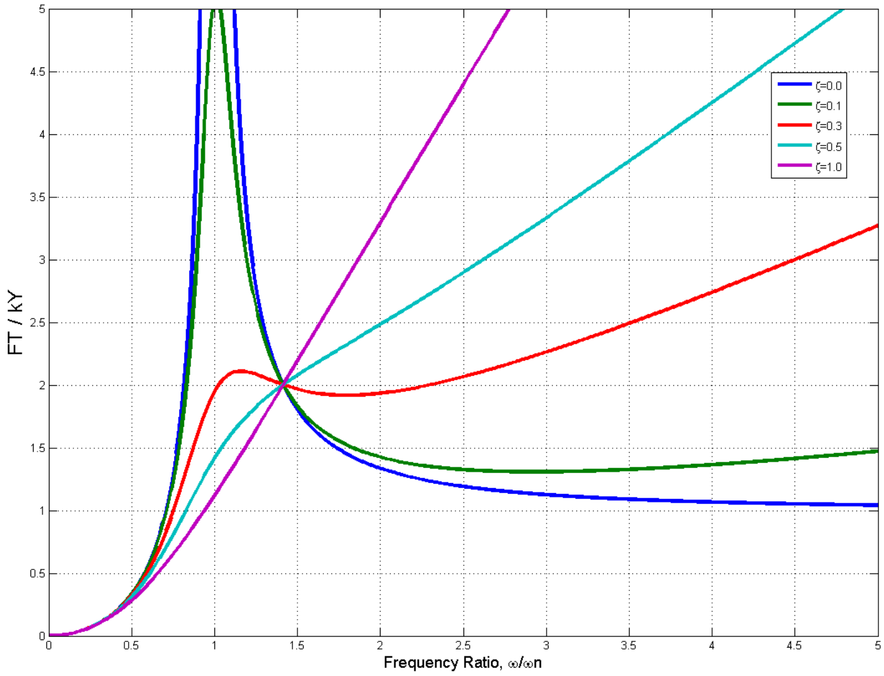
where F_T is the amplitude or maximum value of the force transmitted to the base. Hence:

$$F_T = m\omega^2 Y\frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

from which we get:

$$\frac{F_T}{kY} = r^2\sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}}$$

In this relationship, kY represents the force on the mass if it remained stationary, while the base moved with only the spring attached. This ratio is called the *force transmissibility*.



The force transmitted can also be calculated for the case of a harmonic force (Chapter 9). In this case:

$$F_T = kx + c\dot{x} = kx + i\omega cx$$

so the force amplitude is:

$$|F_T| = \sqrt{(kX)^2 + (\omega cX)^2}$$

Thus, the *transmissibility* or *transmission ratio* of the isolator (T_r) can be calculated to be:

$$T_r = \frac{F_T}{F_0} = \sqrt{\frac{k^2 + \omega^2 c^2}{\left(k - m\omega^2\right)^2 + \omega^2 c^2}} = \sqrt{\frac{1 + (2\zeta r)^2}{\left(1 - r^2\right)^2 + \left(2\zeta r\right)^2}}$$

This has the same amplitude as the base motion ratio found earlier. If T_r is less than one, then the system behaves like a vibration isolator, i.e. the ground receives less force than the input force. (See figure 3.15)

Rotating Unbalance

Having a rotating unbalance is a common problem in machinery. In this case, we have the following equation of motion:

$$M\ddot{x} + c\dot{x} + kx = me\omega^2 \sin \omega t$$

If we denote the particular solution $x_p(t)$ as:

$$x_p(t) = X \sin(\omega t - \phi)$$

then the displacement amplitude is:

$$\frac{X}{X_0} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

where

$$X_0 = \frac{me\omega^2}{k} = \frac{mer^2}{M}$$

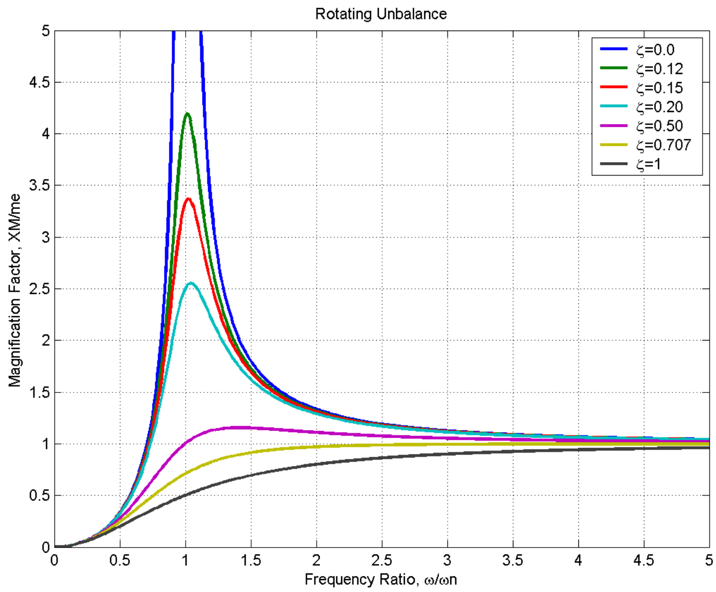
then

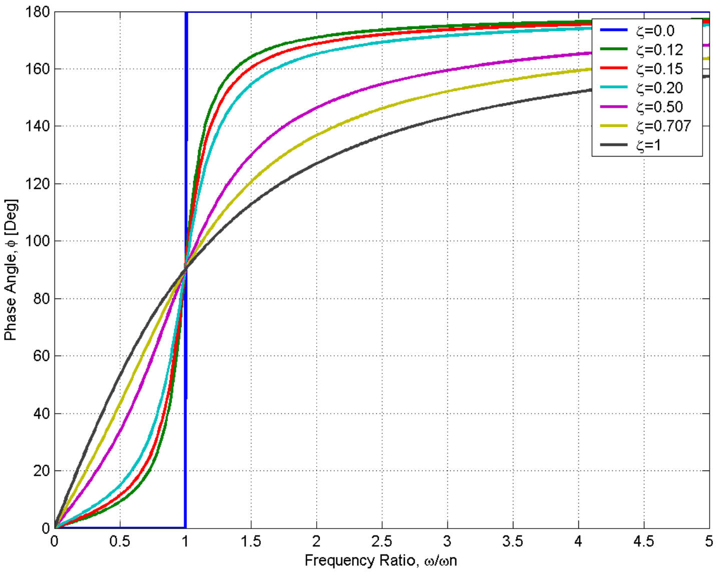
$$\frac{XM}{me} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

and

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$

(See figure 3.17)





(See figure 3.16)

The force transmitted because of the rotating imbalance is

$$F = kx + c\dot{x}$$

The transmissibility can be analyzed as before to give:

$$T_r = \frac{F}{F_0} = \frac{F}{me\omega^2} = \frac{F}{mer^2{\omega_n}^2}$$

thus

$$\frac{F}{me{\omega_n}^2} = r^2\sqrt{\frac{1+(2\zeta r)^2}{\left(1-r^2\right)^2+(2\zeta r)^2}}$$

Response under a general periodic force

If the forcing function is periodic, we can use the Fourier series and the principle of superposition to get the response. The Fourier series states that a periodic function can be represented as a series of sines and cosines:

$$
\begin{aligned}
F(t) &= \frac{a_0}{2} + a_1 \cos \omega t + a_2 \cos 2\omega t + \ldots + b_1 \sin \omega t + b_2 \sin 2\omega t + \ldots \\
&= \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos j\omega t + b_j \sin j\omega t
\end{aligned}
$$

$$a_j = \frac{2}{T}\int_0^T F(t)\cos(j\omega t)\,dt \quad , \quad j = 0, 1, 2, \ldots$$

$$b_j = \frac{2}{T}\int_0^T F(t)\sin(j\omega t)\,dt \quad , \quad j = 1, 2, 3, \ldots$$

where $T = 2\pi/\omega$ is the period. Now the equation of motion can be written as:

$$m\ddot{x} + c\dot{x} + kx = F(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos j\omega t + b_j \sin j\omega t$$

Using the principle of superposition, the steady-sate solution of this equation is the sum of the steady-state solutions of:

$$m\ddot{x} + c\dot{x} + kx = \frac{a_0}{2}$$

$$m\ddot{x} + c\dot{x} + kx = a_j \cos j\omega t$$

$$m\ddot{x} + c\dot{x} + kx = b_j \sin j\omega t$$

The particular solution of the 1st equation is:

$$x_p(t) = \frac{a_0}{2k}$$

The particular solutions of the 2nd and 3rd equations are:

$$x_p(t) = \frac{a_j/k}{\sqrt{\left(1 - j^2 r^2\right)^2 + \left(2\zeta jr\right)^2}} \cos\left(j\omega t - \phi_j\right)$$

$$x_p(t) = \frac{b_j/k}{\sqrt{\left(1 - j^2 r^2\right)^2 + \left(2\zeta jr\right)^2}} \sin\left(j\omega t - \phi_j\right)$$

$$\phi_j = \tan^{-1}\left(\frac{2\zeta jr}{1 - j^2 r^2}\right)$$

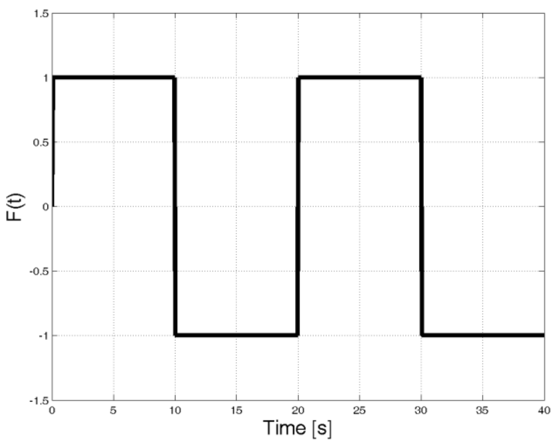
$$r = \frac{\omega}{\omega_n}$$

Then add up all the sums to get the complete steady-state solution as:

$$x_p(t) = \frac{a_o}{2k} + \sum_{j=1}^{\infty} \frac{a_j/k}{\sqrt{\left(1 - j^2 r^2\right)^2 + \left(2\zeta jr\right)^2}} \cos\left(j\omega t - \phi_j\right) + \sum_{j=1}^{\infty} \frac{b_j/k}{\sqrt{\left(1 - j^2 r^2\right)^2 + \left(2\zeta jr\right)^2}} \sin\left(j\omega t - \phi_j\right)$$

Observe that if $j\omega = \omega_n$, the amplitude will be significantly large, especially for small j and ζ. Further, as j becomes large, the amplitude becomes smaller and the corresponding terms tend to zero. How many terms do you need to include?

Example: Obtain the steady-state response of a dynamic system having m=1, c=0.7 and k=1 when subjected to the force shown.



Solution: here we have:

$$T = 20 \quad , \quad \omega = 2\pi/T = 2\pi/20 = \pi/10$$

and the forcing function is given by:

$$F(t) = 1 \qquad 0 \le t \le 10$$

$$F(t) = -1 \quad 10 \le t \le 20$$

The Fourier series of the forcing function is given by:

$$F(t) = \frac{a_0}{2} + a_1 \cos \omega t + a_2 \cos 2\omega t + \ldots + b_1 \sin \omega t + b_2 \sin 2\omega t + \ldots$$

To get the constants, we have:

$$a_0 = \frac{2}{T}\int_0^T F(t)\,dt = \frac{2}{20}\left[\int_0^{10} dt - \int_{10}^{20} dt\right] = 0.1\left[10 - (20-10)\right] = 0$$

$$a_j = \frac{2}{T}\int_0^T F(t)\cos(j\omega t)\,dt$$

$$= \frac{2}{20}\left[\int_0^{10}\cos\left(j\frac{\pi}{10}t\right)dt - \int_{10}^{20}\cos\left(j\frac{\pi}{10}t\right)dt\right]$$

$$= 0.1\left[\frac{10}{j\pi}\sin j\frac{\pi}{10}t\Big|_0^{10} - \frac{10}{j\pi}\sin j\frac{\pi}{10}t\Big|_{10}^{20}\right] = 0$$

i.e. all cosine terms vanish

$$b_j = \frac{2}{T}\int_0^T F(t)\sin(j\omega t)\,dt$$

$$= \frac{2}{20}\left[\int_0^{10}\sin\left(j\frac{\pi}{10}t\right)dt - \int_{10}^{20}\sin\left(j\frac{\pi}{10}t\right)dt\right]$$

$$= 0.1\left[\frac{-10}{j\pi}\cos j\frac{\pi}{10}t\Big|_0^{10} - \frac{-10}{j\pi}\cos j\frac{\pi}{10}t\Big|_{10}^{20}\right]$$

$$= 0.1\left[\frac{-10}{j\pi}(\cos j\pi - 1) + \frac{10}{j\pi}(\cos 2j\pi - \cos j\pi)\right]$$

If j is odd,

$$b_j = 0.1\left[\frac{-10}{j\pi}(-2)+\frac{10}{j\pi}(2)\right]=\frac{4}{j\pi}$$

If j is even,

$$b_j = 0.1\left[\frac{-10}{j\pi}(0)+\frac{10}{j\pi}(0)\right]=0$$

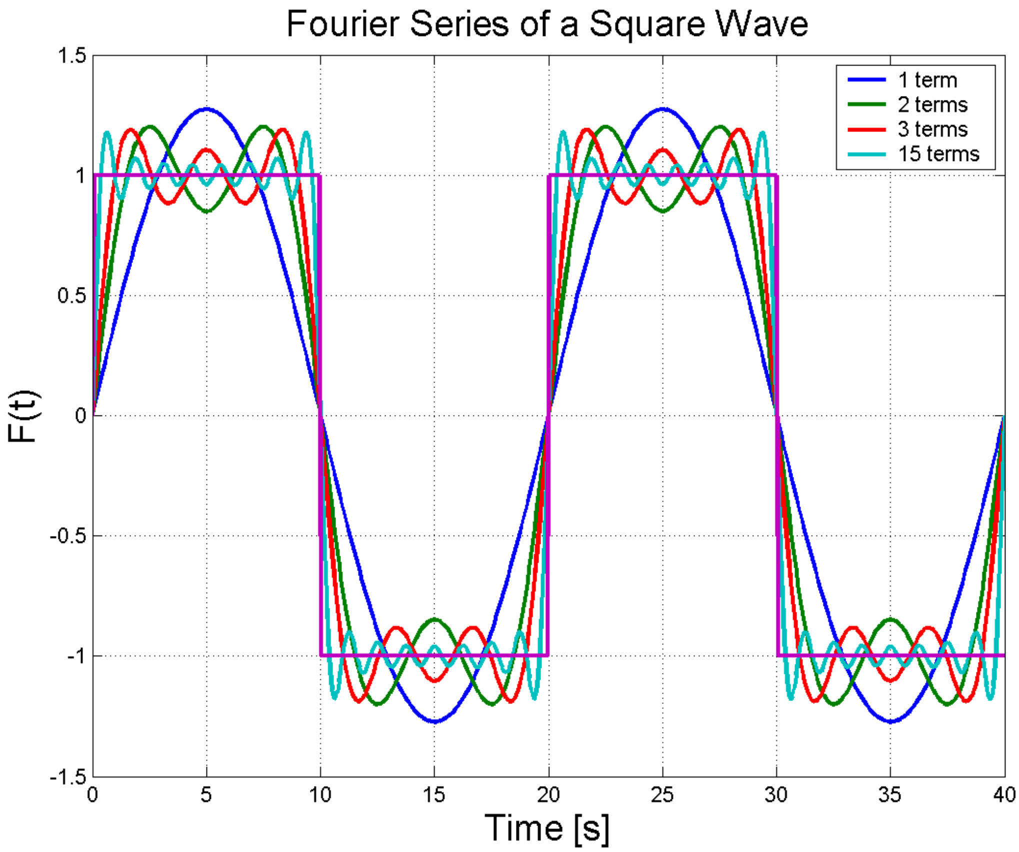
i.e. all even terms vanish.

In this way, the force can be represented by a Fourier series as:

$$F(t)=b_1\sin\omega_1 t+b_3\sin\omega_3 t+b_5\sin\omega_5 t+\cdots=\sum_{j=1}^{\infty}b_j\sin(j\omega t)\quad,\quad j=1,3,5,7,\ldots$$

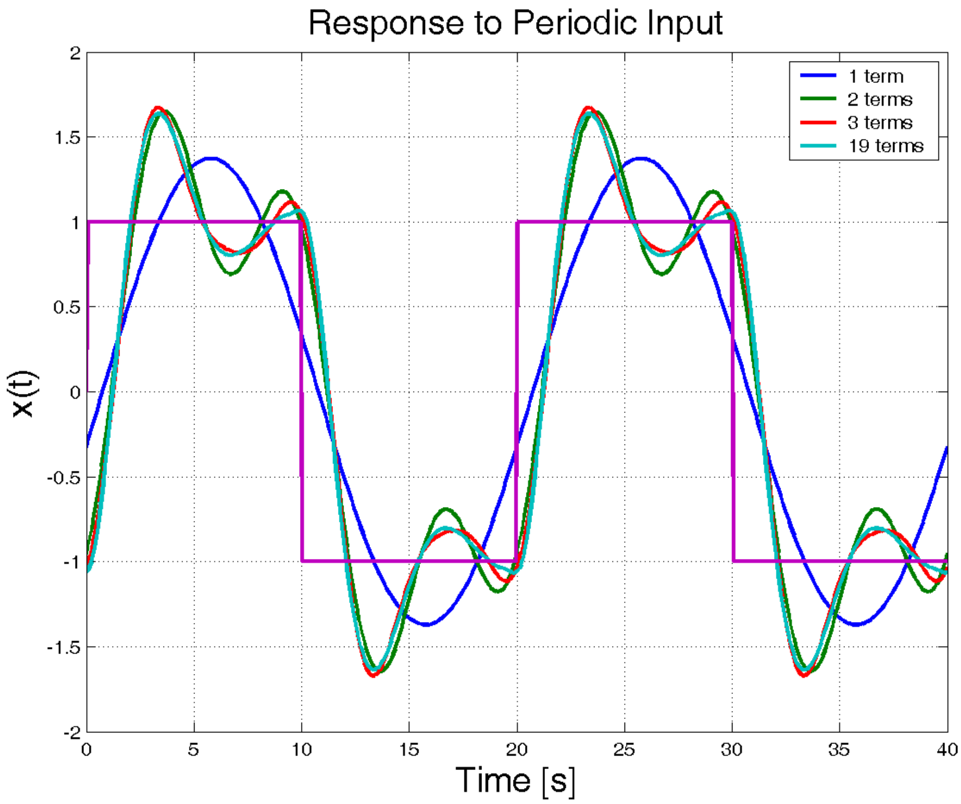
$$=\frac{4}{\pi}\sin\frac{\pi}{10}t+\frac{4}{3\pi}\sin\frac{3\pi}{10}t+\frac{4}{5\pi}\sin\frac{5\pi}{10}t+\cdots=\sum_{j=1}^{\infty}\frac{4}{j\pi}\sin\left(j\frac{\pi}{10}t\right)\quad,\quad j=1,3,5,7,\ldots$$

or graphically as:

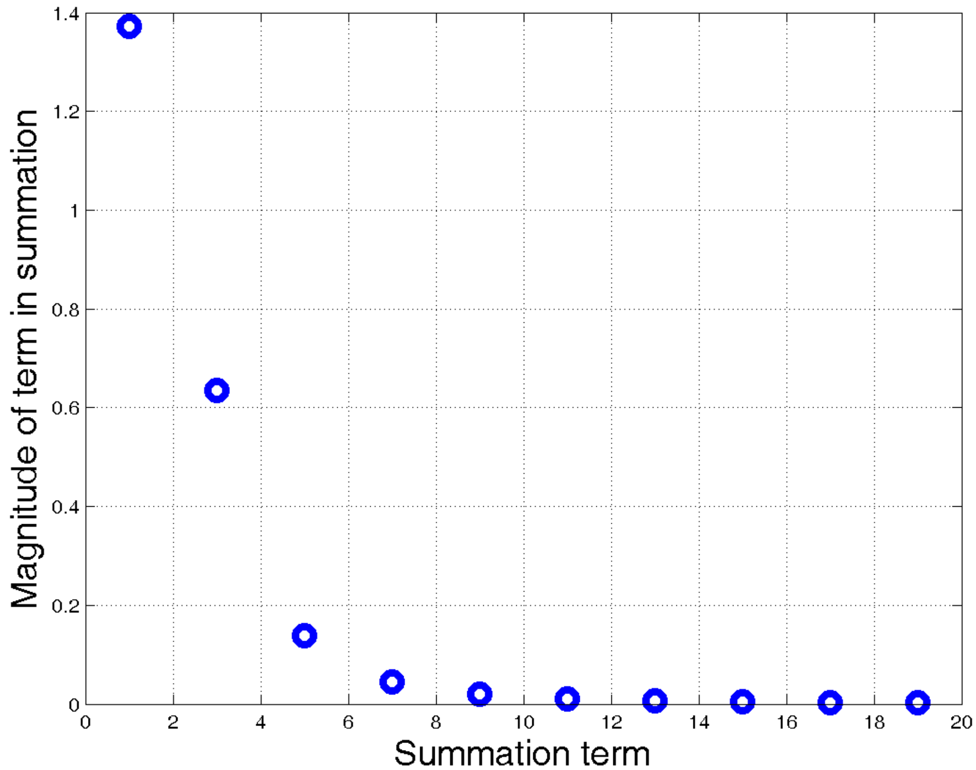


```
%   ********************************************************
%   Periodic response of a dynamic system to a square input waveform
%   ********************************************************
clear; close all;
m=1; c=0.7; k=1; % Parameters

wn=sqrt(k/m); zi=c/wn/2; w=pi/10;

t=linspace(0,40,500); x=zeros(size(t)); r=w/wn;
for jj=1:2:19
    a(jj)=4/pi/jj;
    X(jj)=a(jj)/k/sqrt((1-jj^2*r^2)^2+(2*zi*jj*r)^2);   % term in summation
    phi(jj)=atan2(2*zi*jj*r,1-jj*jj*r*r);
    x(jj+2,:)=x(jj,:)+X(jj)*sin(w*jj*t-phi(jj));
end
u=sign(sin(w*t));

figure(1)
plot(t,x([3 5 7 21],:),t,sign(sin(w*t)),'linewidth',2);grid
legend('1 term','2 terms','3 terms','19 terms')
xlabel('Time [s]','fontsize',18);ylabel('x(t)','fontsize',18)
title('Response to Periodic Input','fontsize',18);

figure(2)
plot(1:2:19,X(1:2:19),'o','markersize',10,'linewidth',4);grid
ylabel('Magnitude of term in summation','fontsize',18)
xlabel('Summation term','fontsize',18)
```

The system response is shown below:



The contribution of each term in the summation is determined from:



Response to an impulse

From ENGR 214, the principle of impulse and momentum states that impulse equals change in momentum:

$$Impulse = F\Delta t = m(v_2 - v_1)$$

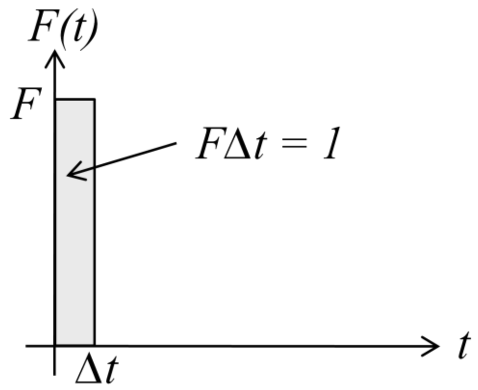
or:

$$\underline{F} = \int_t^{t+\Delta t} F dt = m\dot{x}_2 - m\dot{x}_1$$

A unit impulse is defined as:

$$\underline{f} = \lim_{\Delta t \to 0} \int_t^{t+\Delta t} F dt = 1$$

Now consider the response of an undamped system to a unit impulse. Recall that the free vibration response is given by:



$$x(t) = x_0 cos\omega_n t + \frac{\dot{x}_0}{\omega_n} sin\omega_n t$$

If the mass starts from rest, we can get the velocity just after impulse as:

$$\dot{x}_0 = \frac{\underline{f}}{m} = \frac{1}{m}$$

and the response becomes:

$$x(t) = \frac{1}{m\omega_n} sin\omega_n t$$

for a non-unit impulse, the response is:

$$x(t) = \frac{\underline{F}}{m\omega_n} sin\omega_n t$$

For an underdamped system, recall that the free response was gven by: (see page 142)

$$x(t) = e^{-\zeta\omega_n t}\left[x_0 cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d} sin\omega_d t\right]$$

For a unit impulse, the response for zero initial conditions is:

$$x(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_d} sin\omega_d t = g(t)$$

where $g(t)$ is known as the *impulse response function*. For a non-unit impulse, the response is:

$$x(t) = \frac{\underline{F}e^{-\zeta\omega_n t}}{m\omega_d} sin\omega_d t = \underline{F}g(t)$$

If the impulse occurs at a delayed time $t = \tau$, then

$$x(t) = \frac{\underline{F}e^{-\zeta\omega_n (t-\tau)}}{m\omega_d} sin\omega_d (t-\tau) = \underline{F}g(t-\tau)$$

If two impulses occur at two different times, then their responses will superimpose.

Example: For a system having $m = 1$ kg; $c = 0.5$ kg/s; $k = 4$ N/m; F = 2 Ns obtain the response when two impulses are applied 5 seconds apart.

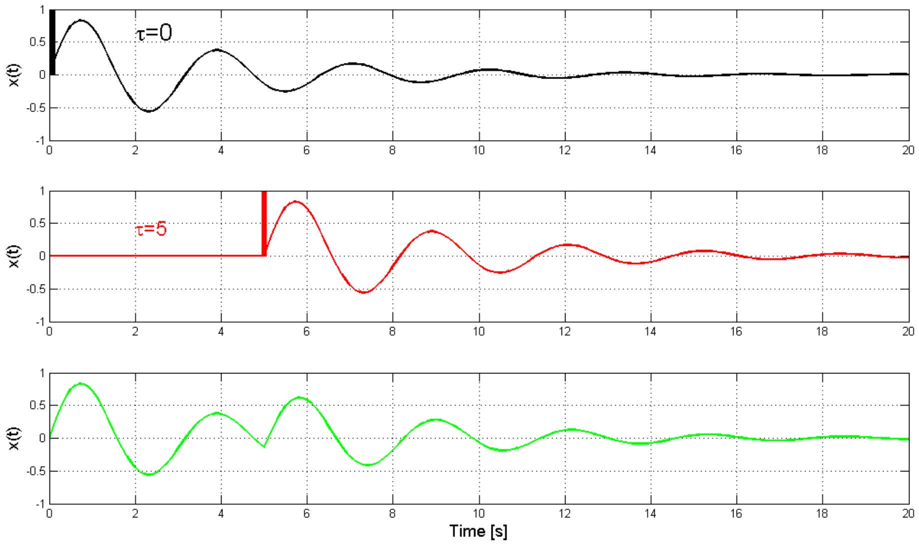
Solution: here we have $\omega_n = 2\frac{\text{rad}}{\text{s}}, \zeta = 0.125, \omega_d = 1.984\frac{\text{rad}}{\text{s}}$ so the solutions become:

$$x_1(t) = \frac{2e^{-0.25t}}{1.984}\sin 1.984t \qquad t > 0$$

$$x_2(t) = \frac{2e^{-0.25(t-\tau)}}{1.984}\sin[1.984(t-\tau)] \qquad t > 5$$

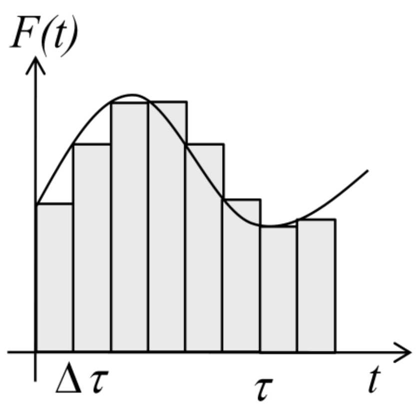
And the total response is:

$$x(t) = \begin{cases} \dfrac{2e^{-0.25t}}{1.984}\sin 1.984t & 0 < t < 5 \\ \dfrac{2e^{-0.25t}}{1.984}\sin 1.984t + \dfrac{2e^{-0.25(t-\tau)}}{1.984}\sin[1.984(t-\tau)] & 5 < t < 20 \end{cases}$$



Response to an Arbitrary Input

The input force is viewed as a series of impulses. The response at time t due to an impulse at time τ is:



$$x(t) = F(\tau)\Delta\tau g(t-\tau)$$

The total response at time t is the sum of all responses:

$$x(t) = \sum F(\tau) g(t-\tau) \Delta\tau$$

Hence

$$x(t) = \int_0^t F(\tau) g(t-\tau) d\tau$$

For an underdamped system:

$$x(t) = \frac{1}{m\omega_d} \int_0^t F(\tau) e^{-\zeta\omega_n(t-\tau)} sin\omega_d(t-\tau)\, d\tau$$

Note: this does not consider initial conditions. This type of formula is called the *convolution integral* or the *Duhamel integral*. For base excitation, the resulting response is

$$z(t) = -\frac{1}{\omega_d} \int_0^t \ddot{y}(\tau) e^{-\zeta\omega_n(t-\tau)} \sin(\omega_d(t-\tau)) d\tau$$

Example: determine the response of a spring-mass-damper system due to the application of a force (see Example 4.6 on page 318).

Here we have

$F(t) = F_0$

so the response is obtained from

$$x(t) = \frac{1}{m\omega_d} \int_0^t F_0 e^{-\zeta\omega_n(t-\tau)} \sin\left[\omega_d(t-\tau)\right] d\tau$$

You can integrate this by parts, or look it up in a table of integrals. Here we will use MATLAB to symbolically integrate the equation. In MATLAB we write

```
syms z wn t tau wd
M=int(exp(-z*wn*(t-tau))*sin(wd*(t-tau)),tau,0,t)

M =
-(-wd+exp(-z*wn*t)*wd*cos(wd*t)+exp(-
z*wn*t)*z*wn*sin(wd*t))/(z^2*wn^2+wd^2)

pretty(M)
         -wd + exp(-z wn t) wd cos(wd t) + exp(-z wn t) z wn sin(wd t)
       - -------------------------------------------------------------------------
                              2   2     2
                             z  wn  + wd
```

Therefore the solution is:

$$x(t) = \frac{-F_0}{m\omega_d}\left[\frac{-\omega_d + e^{-\zeta\omega_n t}\omega_d \cos\omega_d t + e^{-\zeta\omega_n t}\zeta\omega_n \sin\omega_d t}{\zeta^2\omega_n^2 + \omega_d^2}\right]$$

which can be put in the form:

$$x(t) = \frac{-F_0}{m\omega_d}\left[\frac{-\omega_n\sqrt{1-\zeta^2} + e^{-\zeta\omega_n t}\omega_n\sqrt{1-\zeta^2}\cos\omega_d t + e^{-\zeta\omega_n t}\zeta\omega_n \sin\omega_d t}{\zeta^2\omega_n^2 + \omega_n^2\left(1-\zeta^2\right)}\right]$$

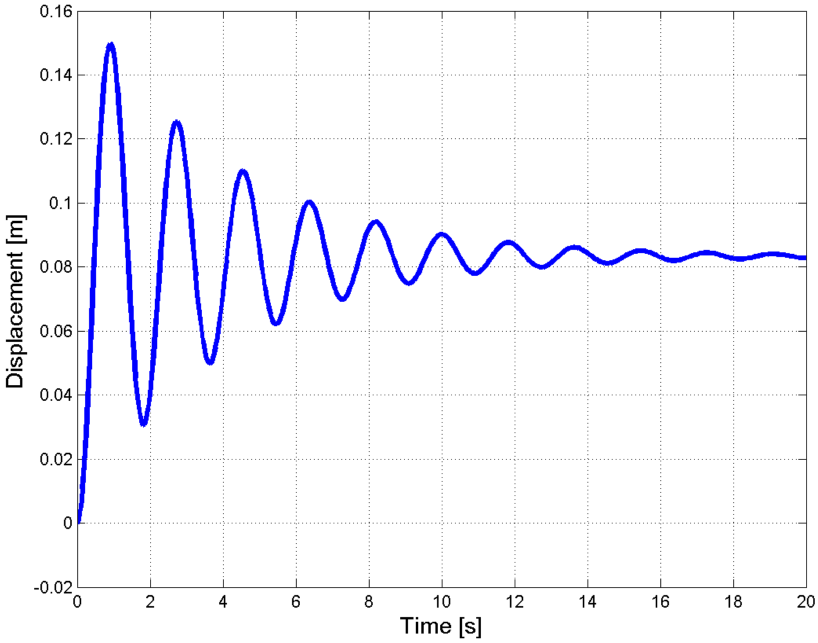
$$= \frac{-F_0}{m\omega_d\omega_n}\left[-\sqrt{1-\zeta^2} + e^{-\zeta\omega_n t}\cos\left(\omega_d t - \phi\right)\right]$$

$$= \frac{F_0}{k}\left[1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos\left(\omega_d t - \phi\right)\right]$$

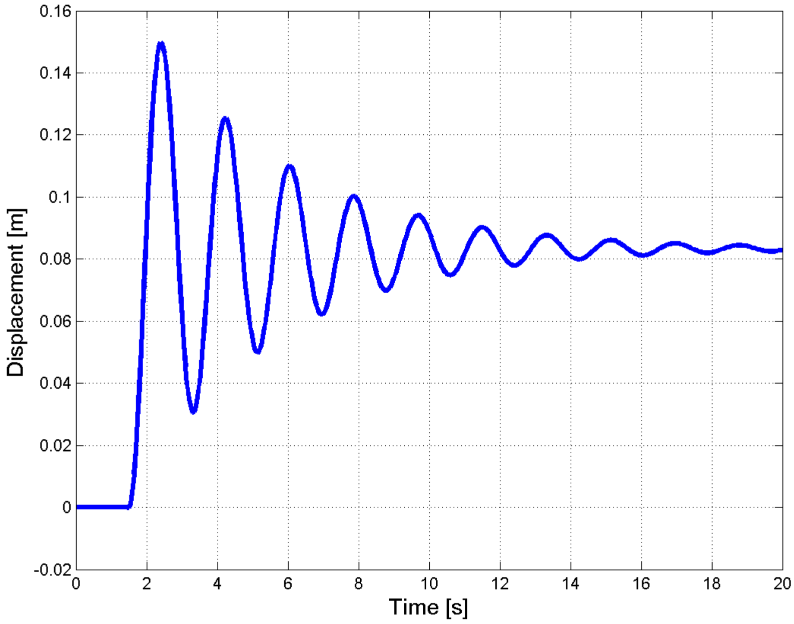
where

$$\phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

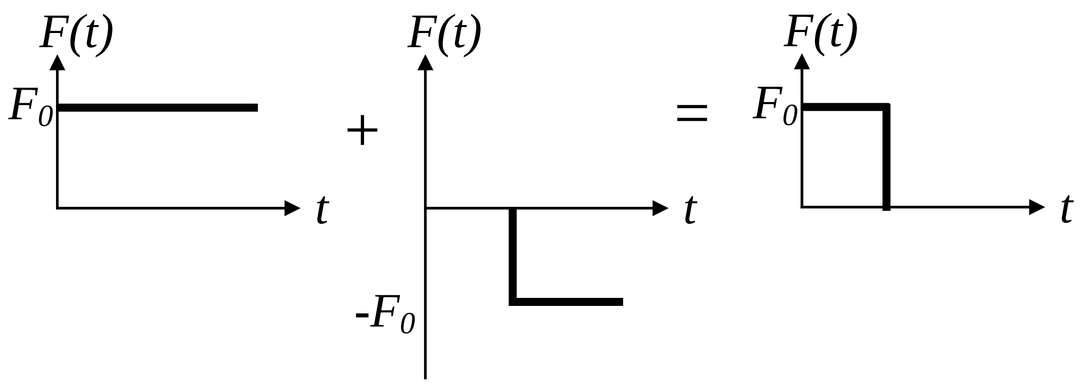
The response is shown below. Notice how it converges to $\frac{F_0}{k}$.



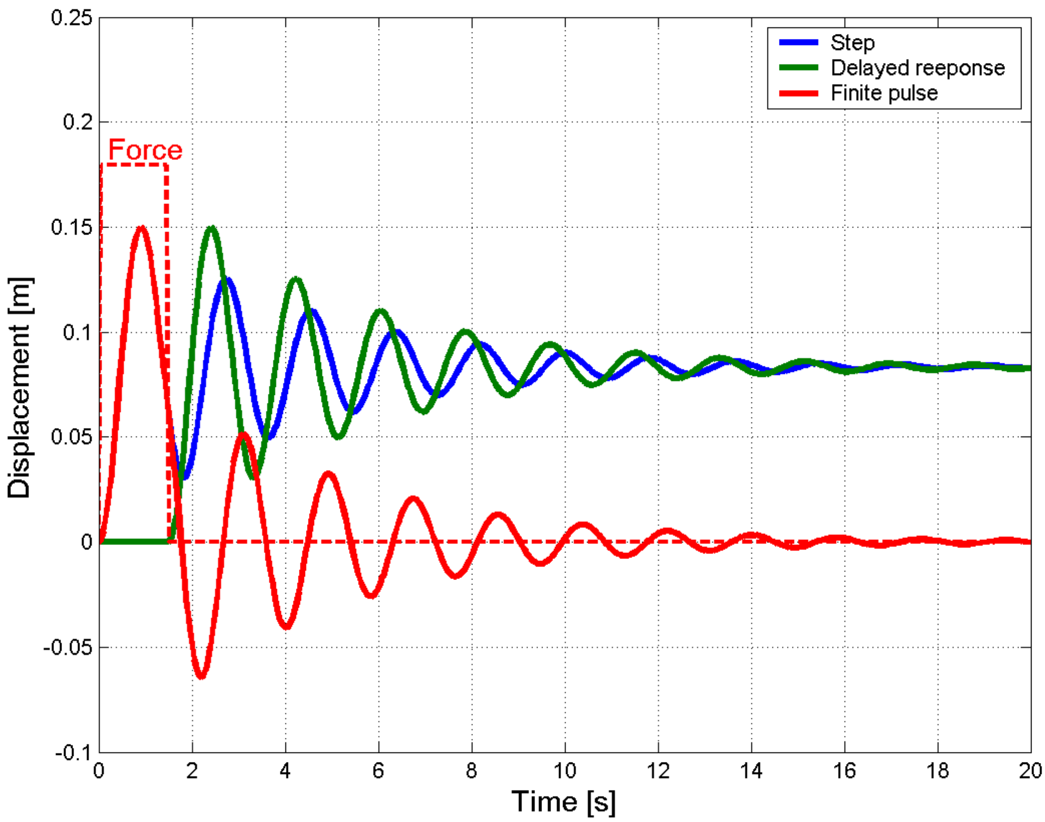
Example: delayed step force (see Example 4.7, page 319). The solution is obtained directly from MATLAB by replacing the time vector t with a new t2, where t2=t-t0. Note that from t=0 to t=t0 no force exists, and hence no displacement should be present, so you have to impose zero displacement. The response is



Example: Pulse with finite width (see Example 4.8, page 320)



Solution: The given forcing function can be considered a sum of 2 step functions. Thus the total response is the sum of the 2 responses mentioned earlier.



See figure 4.9.

If there is no damping (c=0) some interesting things can happen

$$x(t) = x_0 \cos(\omega_n t) + \frac{\dot{x}_0}{\omega_n}\sin\left(\omega_n t\right) + \delta_{st}\frac{\cos(\omega t) - \cos(\omega_n t)}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$

As the forced frequency approaches the natural frequency ($\omega \rightarrow \omega_n$)

$$\lim_{\omega\to\omega_n}\frac{\cos(\omega t) - \cos(\omega_n t)}{1-\left(\frac{\omega}{\omega_n}\right)^2} = \lim_{\omega\to\omega_n}\frac{\frac{d}{d\omega}\left(\cos(\omega t) - \cos(\omega_n t)\right)}{\frac{d}{d\omega}\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)}$$

$$= \lim_{\omega\to\omega_n}\left(\frac{-t\sin\omega t}{-2\frac{\omega}{\omega_n^{\ 2}}}\right) = \frac{\omega t}{2}\sin\omega t$$

so

$$x(t) = x_0 \cos(\omega_n t) + \frac{\dot{x}_0}{\omega_n}\sin\left(\omega_n t\right) + \delta_{st}\frac{\omega t}{2}\sin\left(\omega t\right)$$

See Fig 3.6

If the frequencies are close ($\omega \approx \omega_n$) then a phenomena called beating occurs. Assuming zero initial conditions

$$x(t) = \delta_{st}\frac{\cos(\omega t) - \cos(\omega_n t)}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$

$$= \frac{F_0}{m\left(\omega_n^{\ 2} - \omega^2\right)}2\sin\left(\frac{\omega+\omega_n}{2}t\right)\sin\left(\frac{\omega_n-\omega}{2}t\right)$$

Using the following notation

$$\omega_n - \omega = 2\varepsilon \quad \omega_n + \omega \approx 2\omega \quad \omega_n^{\ 2} - \omega^2 = 4\varepsilon\omega$$

then

$$x(t) = \frac{\delta_{st}}{2\omega\varepsilon}\sin(\varepsilon t)\sin(\omega t)$$

This will be a sin wave with a slowly varying sinusoidal magnitude.

Forced vibration with Coulomb Damping

For the system

$$m\ddot{x} \pm \mu N + kx = F_0 \cos\omega t$$

the solution is

$$x_p(t) = X\cos(\omega t - \phi)$$

$$X = \frac{F_0}{k}\sqrt{\frac{1-\left(\frac{4\mu N}{\pi F_0}\right)^2}{\left(1-\frac{\omega^2}{\omega_n{}^2}\right)^2}} \qquad \phi = \tan^{-1}\left(\frac{\pm\frac{4\mu N}{\pi F_0}}{1-\frac{\omega^2}{\omega_n{}^2}}\right)$$

which is valid for $F_0 >> \mu N$

Forced vibration with Hysteretic Damping

For the system

$$m\ddot{x} \pm \frac{bk}{\omega}\dot{x} + kx = F_0 \cos\omega t$$

the solution is

$$x_p(t) = X\cos(\omega t - \phi)$$

$$X = \frac{\frac{F_0}{k}}{\sqrt{\left(1-\frac{\omega^2}{\omega_n{}^2}\right)^2 + \beta^2}} \qquad \phi = \tan^{-1}\left(\frac{\beta}{1-\frac{\omega^2}{\omega_n{}^2}}\right)$$

See figure 3.24