MENG 445
Forecasting- the constant model
Moving Average (MA) and Exponential Smoothing (ES)

For the constant model: $X_t = a + e_t$, $e_t \approx NID(0, \sigma^2)$, we used the moving average M_T defined as:
$M_T = \sum_{t=T-N+1}^{T} \frac{X_t}{N}$, to forecast any future period such that: $\hat{x}_{T+\tau}(T) = M_T$ for $\tau \geq 1$

We also defined the forecasting error e_T as: $e_T = X_T - M_{T-1}$

There are two problems with M_T:
1- All observations are assigned equal weight.
2- The forecast is not adjusted by the amount of error.

To avoid these problem we propose a heuristic approach where we define:
current forecast = previous forecast + percentage of current forecasting error.

Using S to denote the forecast we can write:

$$S_T = S_{T-1} + \alpha(X_T - S_{T-1}),\ 0<\alpha<1 \qquad (1)$$

So that, the forecast at current time T is equal to the forecast made at the previous time period + a percentage of the error in the previous forecast.

Using S_T, the forecast for any future interval is giving by: $\hat{x}_{T+\tau}(T) = S_T$

Notice that equation (1) could be written as:

$$S_T = \alpha x_T + (1-\alpha)S_{T-1} \qquad (2)$$

Following the logic of equation (2) we could write

$$S_{T-1} = \alpha X_{T-1} + (1-\alpha)S_{T-2} \qquad (3)$$

Substituting (3) into (1) we get:

$$S_T = \alpha X_T + (1-\alpha)[\alpha X_{T-1} + (1-\alpha)S_{T-2}]$$
$$S_T = \alpha X_T + \alpha(1-\alpha)X_{T-1} + (1-\alpha)^2 S_{T-2}$$

Therefor, we could write

$$S_T = \alpha \sum_{K=0}^{T-I} (1-\alpha)^K X_{T-k} + (1-\alpha)^T S_0 \qquad (4)$$

Using equation 4 we can easily show that S_T is an unbiased estimate for the process level; that is:
$\underset{T\to\infty}{E}\left[S_T\right] = a$.

We also can easily show that $V\left[S_T\right] = \frac{\alpha}{2-\alpha}\sigma^2$

<u>I. Summary for the Constant Model</u>

$$x_t = a + \varepsilon_t,\ \varepsilon_t \approx NID(0, \sigma^2)$$

a. Moving Average procedure (same as Least Square)

$$M_T = \frac{1}{N}\sum_{t=T-N+1}^{T} x_t, \quad V(M_T) = \frac{\sigma^2}{N}, \quad \hat{x}_{T+\tau}(T) = M_T \pm 2\sqrt{1+\frac{1}{N}}\sigma$$

b. Exponential Smoothing

$$S_T = \alpha x_T + (1-\alpha) S_{T-1}, \quad V(S_T) = \frac{\alpha}{2-\alpha}\sigma^2, \quad \hat{x}_{T+\tau}(T) = S_T \pm 2\sigma\sqrt{1+\frac{\alpha}{2-\alpha}}$$

Example

Assuming that the following data follows the constant model:

$X_t = a + e_t,\ e_t \approx NID(0, 2.25)$,

Provide a 95% prediction interval on the demand in period 11. Use either N = 3 or N = 4; whichever minimizes some measure of error.

		MOVING AVERAGE (N=3)				MOVING AVERAGE (N=4)			
t	x_t	M_t	$\hat{x}_t(t-1)$	e_t	$\lvert e_t \rvert$	M_t	$\hat{x}_t(t-1)$	e_t	$\lvert e_t \rvert$
0									
1	10.00								
2	12.00								
3	11.00	11.00							
4	10.00	11.00	11.00	-1.00	1.00	10.75			
5	9.00	10.00	11.00	-2.00	2.00	10.50	10.75	-1.75	1.75
6	11.00	10.00	10.00	1.00	1.00	10.25	10.50	0.50	0.50
7	11.00	10.33	10.00	1.00	1.00	10.25	10.25	0.75	0.75
8	12.00	11.33	10.33	1.67	1.67	10.75	10.25	1.75	1.75
9	10.00	11.00	11.33	-1.33	1.33	11.00	10.75	-0.75	0.75
10	12.00	11.33	11.00	1.00	1.00	11.25	11.00	1.00	1.00
			11.33				11.25		

	N = 3	N = 4
MAD	1.29	1.08
MSE	1.79	1.42

MA, N = 4 $\rightarrow$ $7.90 < \hat{x}_{11} < 14.60$

Apply the ES method to the same data set using an α value that is equivalent to the optimum N in part a.

		MOVING AVERAGE (N=4)				EXPONENTIAL SMOOTHING (α = 0.4)			
t	x_t	M_t	$\hat{x}_t(t-1)$	e_t	$\lvert e_t \rvert$	S_t	$\hat{x}_t(t-1)$	e_t	$\lvert e_t \rvert$
0									
1	10.00								
2	12.00								
3	11.00								
4	10.00	10.75				10.75			
5	9.00	10.50	10.75	-1.75	1.75	10.05	10.75	-1.75	1.75
6	11.00	10.25	10.50	0.50	0.50	10.43	10.05	0.95	0.95
7	11.00	10.25	10.25	0.75	0.75	10.66	10.43	0.57	0.57
8	12.00	10.75	10.25	1.75	1.75	11.19	10.66	1.34	1.34
9	10.00	11.00	10.75	-0.75	0.75	10.72	11.19	-1.19	1.19
10	12.00	11.25	11.00	1.00	1.00	11.23	10.72	1.28	1.28
			11.25				11.23		

	MA, N = 4	ES, α = 0.4	ES, α = 0.5
MAD	1.08	1.18	1.11
MSE	1.42	1.53	1.36

ES, α = 0.4 → $7.88 < \hat{x}_{11} < 14.58$