Quantized vs. Analog Channel Feedback for FDD Massive MIMO Systems with Multiple-Antenna Users

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***Abstract*—In this paper, we consider the problem of channel feedback in massive multiple-input-multiple-output (MIMO) systems. For the downlink scenario, we present a detailed comparison between the performance of the quantized and the analog channel feedback schemes for the case of having multiple antenna users. Both schemes' performance is evaluated by deriving an upper bound on the rate gap between the rate of the system with perfect channel state information (CSI) and with imperfect CSI for both feedback schemes. We compare the two schemes, namely, quantized channel feedback and analog channel feedback, under the same resources allocated for channel feedback for a fair comparison. Moreover, we consider two different downlink transmission schemes; the first one does not consider power allocation across the streams and the second one does power allocation (water-filling) across the streams. Our results show that the analog feedback scheme performs better in the low signal to noise (SNR) region when performing power allocation across the multiple data streams. However, the quantized channel feedback scheme performs better at the high SNR region, where the quantized CSI can provide a better approximation of the actual CSI. Finally, simulation results are presented to verify our theoretical analysis and demonstrate our conclusions.**

I. Introduction

Massive multiple-input multiple-output (MIMO) wireless cellular systems introduced substantial developments on both the spectral and energy efficiency aspects. These gains are achievable using simple linear precoding techniques at the Base-Station (BS) to serve multiple users in the system [1], [2]. The downlink channel state information (CSI) must be known at the massive MIMO BS to apply data precoding and beamforming techniques, hence, fully utilizing the array gains of massive MIMO systems. This, in turn, poses a challenge for the practicality of massive MIMO systems. The problem of obtaining the downlink CSI at the BS in frequency division duplex (FDD) systems is challenging because the channel reciprocity cannot be exploited to obtain the downlink CSI from the uplink CSI as in the case of time division duplex (TDD) systems. In FDD massive MIMO systems, the downlink channels are estimated at the user and then transmitted back to the BS. However, estimating the downlink CSI leads to overwhelming feedback overhead as the overhead increases linearly with the number of BS antennas, which affects the system bandwidth. Hence, many papers in the literature have focused on reducing the feedback overhead in FDD massive MIMO systems without causing performance degradation due to this reduction. The quality of the CSI feedback at the BS plays a critical role in improving the system performance.

In [3], the authors presented a spatially common sparsity-based adaptive channel estimation and feedback in FDD massive MIMO systems. They further evaluated their proposed scheme by deriving the Cramér-Rao bound. In [4], the authors proposed a novel feedback reduction scheme using principal component analysis (PCA), which utilizes the spatial correlation among the massive MIMO channels using a compression matrix. In [5], a robust closed-loop pilot and CSI feedback resource adaptation framework was introduced. The authors utilized the joint sparsity of the massive MIMO users' channels to enhance the CSI quality. The authors is [6] proposed a low-dimensional subspace codebook for millimeter-wave (mmWave) massive MIMO systems relying on lens antenna array. In [7], a real-time CSI feedback scheme based on deep learning was proposed for time-varying massive MIMO channels. The authors in this work studied the trade-off between the feedback compression ratio and the proposed scheme's complexity. In [8], the authors proposed an angle of departure (AoD) adaptive codebook for channel feedback in massive MIMO systems. They derived bounds for the rate reduction to evaluate the performance of their scheme. However, the authors did not consider the general case of serving multiple antenna users and made their derivations for the single-antenna users' special case.

In this paper, we introduce a detailed study on the impact of both quantized and analog channel feedback on the performance of AoD based massive MIMO systems with multiple antenna users. To the best of our knowledge, studying this problem for massive MIMO systems with multiple-antenna users was not studied before. A detailed analysis of the performance degradation due to both quantized and analog channel feedback is introduced. We derive bounds on the rate gap between ideal CSI and imperfect CSI at the BS for both quantized and analog feedback schemes. Then, the two schemes are compared against each other using the same feedback resources for a fair comparison. We will show that for the case of limited feedback resources, the analog feedback-based scheme outperforms the quantized one; however, for having more available feedback resources, the quantized feedback-based scheme results in better performance compared to the analog one. Additionally, the analog channel feedback scheme's performance is improved using power allocation across the multiple data streams in the low signal to noise ratio (SNR) region. Moreover, we show that the analog feedback-based scheme outperforms the quantized one in the low SNR region; however, the quantized feedback-based scheme overtakes in the high SNR region.

II. System Model

A. Downlink Massive MIMO Channel Model

The system model in this paper contains a mmWave massive MIMO BS with M antennas that communicates with K multiple antenna users through the downlink channel. The kth user, $\forall k \in \{1, 2, \cdots, K\}$, has N_k receiving antennas. The number of transmitting antennas at the BS is assumed to be much higher than the number of users' antennas (i.e., $M >> \sum_k N_k$). The channel matrix of the kth user, $\mathbf{H}_k \in \mathbb{C}^{N_k \times M}$, is formed using the downlink narrowband ray-based channel model in [8], [9], and it can be expressed as

$$\mathbf{H}_k = \mathbf{G}_k \mathbf{A}_k(\phi_{k,1}, \phi_{k,2}, \ldots, \phi_{k,P_k}). \tag{1}$$

The matrix $\mathbf{A}_k(\phi_{k,1}, \phi_{k,2}, \ldots, \phi_{k,P_k}) \in \mathbb{C}^{P_k \times M}$ is defined as

$$\mathbf{A}_k(\phi_{k,1}, \phi_{k,2}, \ldots, \phi_{k,P_k}) = \left[\mathbf{a}(\phi_{k,1})\ \mathbf{a}(\phi_{k,2})\ \ldots\ \mathbf{a}(\phi_{k,P_k})\right]^T \tag{2}$$

where P_k is the number of resolvable paths from the BS to the kth user, and $\phi_{k,i}(1 \le i \le P_k)$ is the AoDs of the i^{th} path of the kth user. The transmit antennas at the BS form a uniform linear array (ULA), so $\mathbf{a}(\phi_{k,i}) \in \mathbb{C}^{M \times 1}$ is the antenna response vector of the i^{th} propagation path of user k, and it can be expressed as

$$\mathbf{a}(\phi_{k,i}) = \left[1, e^{-j2\pi \frac{d}{\lambda} \sin(\phi_{k,i})}, \ldots, e^{-j2\pi \frac{d}{\lambda}(M-1)\sin(\phi_{k,i})}\right]^T, \tag{3}$$

where λ is the signal wavelength, d is the distance between the antennas at the BS. The entry $\mathbf{G}_k(i,j)$ is the complex path gain of the j^{th} path of the i^{th} antenna at user k. The complex path gains in $\mathbf{G}_k$ are independently and identically distributed (i.i.d.) complex normal random variables with zero mean and unit variance. It is noted from (1) that the channel vector of each antenna of user k is a linear combination of its P_k steering vectors scaled by the complex path gains of that antenna.

Within the coherence period of $\phi_{k,i}$, the channel vector of every antenna of user k is distributed in a subspace of dimension P_k of the full M-dimensional space. This subspace is called the channel subspace. The channel subspace, $\mathbf{A}_k$, is a function of the AoDs which are assumed to be known at the massive BS and the kth user. This assumption is valid as the kth user can estimate the AoDs applying the multiple signal classification (MUSIC) algorithm [10]. After that, user k feeds back the AoDs to the massive BS every angle coherence period so that the BS can generate $\mathbf{A}_k$. As long as the massive BS knows the channel subspace, user k needs to only feedback the low dimensional path gain matrix $\mathbf{G}_k \in \mathbb{C}^{N_k \times P_k}$, and hence, the BS can generate the channel matrix of the kth user $\mathbf{H}_k$. Throughout this work, we assume that the BS has perfect knowledge of the AoDs as their feedback overhead is negligible compared to the feedback overhead of the path gains matrix $\mathbf{G}_k$.

We assume that there are N_k data streams that are sent from the massive BS to user k which are represented by the data vector $\mathbf{m}_k \in \mathbb{C}^{N_k \times 1}$. The data symbol vector of user k is first multiplied by the precoding matrix, $\mathbf{V}_k \in \mathbb{C}^{M \times N_k}$, before transmission. Then, the BS adds the precoded data vectors of all users forming the overall vector, $\mathbf{x} \in \mathbb{C}^{M \times 1}$, that is broadcast to all users and it is written as

$$\mathbf{x} = \sum_{j=1}^{K} \mathbf{V}_j \mathbf{m}_j, \tag{4}$$

and the received signal at user k is given as

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x} + \mathbf{n}_k = \mathbf{H}_k \mathbf{V}_k \mathbf{m}_k + \mathbf{H}_k \sum_{j=1, j \neq k}^{K} \mathbf{V}_j \mathbf{m}_j + \mathbf{n}_k, \tag{5}$$

where $\mathbf{n}_k \in \mathbb{C}^{N_k \times 1}$ is the noise vector at user k whose elements are i.i.d. complex normal random variables with zero mean and unit variance.

The second term in (5) represents the interference from all other signals, $\mathbf{m}_j, j \neq k$, at user k. As long as the precoding matrix, $\mathbf{V}_k$, is unitary (i.e., $\mathbf{V}_k^{\text{H}} \mathbf{V}_k = \mathbf{I}_{N_k}$), then the average squared norm of the data vector of user k is set as $E\left[\|\mathbf{m}_k\|^2\right] = \frac{\gamma}{K}, \forall k \in \{1, 2, \cdots, K\}$, where γ is the total transmit power constraint at the BS.

B. The Per-User Rate

The precoding technique that is used in this paper is the zero-forcing block diagonalization (BD), which was designed for massive MIMO systems with multiple antenna users in [9]. In BD precoding, only the spatial direction of the channel matrix, $\widetilde{\mathbf{H}}_k$, is needed at the BS. The matrix, $\widetilde{\mathbf{H}}_k$, is unitary and its row-space is the row-space of $\mathbf{H}_k$. Since the BS knows $\mathbf{A}_k$, the kth user performs subspace quantization on $\mathbf{G}_k$ [9], then it feeds back the quantized path gains subspace $\widehat{\mathbf{G}}_k$. The BD strategy involves linear precoding that eliminates the inter-user interference. Hence, the second term in (5) is canceled in the case of perfect CSI at the BS, i.e., $\widehat{\mathbf{G}}_k \equiv \widetilde{\mathbf{G}}_k$, where $\widetilde{\mathbf{G}}_k$ is unitary and its row-space is the row-space of $\mathbf{G}_k$. Then, the per-user Ergodic rate for the ideal CSI case is given by [11], [12]

$$R_{\text{Ideal}}(\gamma) = \mathbb{E} \log_2 \left| \mathbf{I}_N + \frac{\gamma}{K N_k} \mathbf{H}_k \mathbf{V}_k \mathbf{V}_k^{\text{H}} \mathbf{H}_k^{\text{H}} \right|. \tag{6}$$

In the case of subspace quantized channel feedback, the interference at the kth user due to all other users cannot be completely eliminated because the row-space of the quantized channel matrix, $\widehat{\mathbf{H}}_k$, is an approximation to the the original spatial direction, $\widetilde{\mathbf{H}}_k$. As a result, this quantization leads to some residual interference power, and the per-user rate is hence given by [13]

$$R_{\text{QUANT}} = \mathbb{E}\log_2\left|\mathbf{I}_N + \frac{\gamma}{KN_k}\sum_{j=1}^{K}\mathbf{H}_k\widehat{\mathbf{V}}_j\widehat{\mathbf{V}}_j^{\text{H}}\mathbf{H}_k^{\text{H}}\right| - \mathbb{E}\log_2\left|\mathbf{I}_N + \frac{\gamma}{KN_k}\sum_{j=1,j\neq k}^{K}\mathbf{H}_k\widehat{\mathbf{V}}_j\widehat{\mathbf{V}}_j^{\text{H}}\mathbf{H}_k^{\text{H}}\right|. \quad (7)$$

where the expectation is evaluated over the distribution of the channel matrix, $\mathbf{H}_k$, and the corresponding quantized precoding matrices, $\widehat{\mathbf{V}}_j$. The term $\mathbf{H}_k\widehat{\mathbf{V}}_k\widehat{\mathbf{V}}_k^{\text{H}}\mathbf{H}_k^{\text{H}}$ represents the useful signal intended for the kth use while $\sum_{j=1,j\neq k}^{K}\mathbf{H}_k\widehat{\mathbf{V}}_j\widehat{\mathbf{V}}_j^{\text{H}}\mathbf{H}_k^{\text{H}}$ represents the inter-user interference at user k.

III. Rate gap of quantized channel Feedback

In this section, we evaluate the rate gap between the ideal system with perfect CSI and the rate of a practical system with quantized channel matrices at the BS. The rate gap is calculated assuming that all users have the same number of receive antennas, i.e., $N_k = N$.

A. Rate gap calculations

The per-user rate of the ideal and quantized CSI at the BS are given by (6) and (7) respectively. Following Theorem 1 of [13], that obtains an upper bound for the quantized rate gap in Multi-User MIMO systems, the per-user rate gap, $\Delta R_{\text{QUANT}} = R_{\text{Ideal}} - R_{\text{QUANT}}$, in our massive MIMO system can be bounded as

$$\Delta R_{\text{QUANT}} \leq \log_2\left|\mathbf{I}_N + \frac{\gamma MP}{K}(K-1)\mathbb{E}\left[\widetilde{\mathbf{H}}_k\widehat{\mathbf{V}}_j\widehat{\mathbf{V}}_j^{\text{H}}\widetilde{\mathbf{H}}_k^{\text{H}}\right]\right| \quad (8)$$

The expectation $\mathbb{E}\left[\widetilde{\mathbf{H}}_k\widehat{\mathbf{V}}_j\widehat{\mathbf{V}}_j^{\text{H}}\widetilde{\mathbf{H}}_k^{\text{H}}\right]$ is evaluated as follows. First, the channel subspace, $\widetilde{\mathbf{H}}_k$, is decomposed as lemma 1 in [13] as

$$\widetilde{\mathbf{H}}_k = \mathbf{R}_k\mathbf{F}_k\widehat{\mathbf{H}}_k + \mathbf{Z}_k\mathbf{S}_k, \quad (9)$$

where $\mathbf{F}_k \in \mathbb{C}^{N\times N}$ is unitary and uniformly distributed over the Grassmannian manifold $\mathcal{G}_{N,N}$, $\mathbf{Z}_k \in \mathbb{C}^{N\times N}$ is lower triangular with positive diagonal elements and satisfies $\text{tr}(\mathbf{Z}_k\mathbf{Z}_k^{\text{H}}) = d^2(\mathbf{H}_k, \widehat{\mathbf{H}}_k)$, $\mathbf{R}_k \in \mathbb{C}^{N\times N}$ is lower triangular with positive diagonal elements satisfying $\mathbf{R}_k\mathbf{R}_k^{\text{H}} = \mathbf{I}_N - \mathbf{Z}_k\mathbf{Z}_k^{\text{H}}$, and the rows of $\mathbf{S}_k \in \mathbb{C}^{N\times M}$ form an orthonormal basis for an isotropically distributed complex N dimensional subspace in the $M-N$ dimensional right nullspace of $\widehat{\mathbf{H}}_k$. Moreover, the matrices $\mathbf{R}_k$, $\widehat{\mathbf{H}}_k$ and $\mathbf{F}_k$ are independent of each other, as are the pair $\mathbf{Z}_k$ and $\mathbf{S}_k$. By right multiplying both sides of (9) by $\widehat{\mathbf{V}}_j$, we get

$$\widetilde{\mathbf{H}}_k\widehat{\mathbf{V}}_j = \mathbf{Z}_k\mathbf{S}_k\widehat{\mathbf{V}}_j, \quad (10)$$

for $k \neq j$ due to the fact that $\widehat{\mathbf{H}}_k\widehat{\mathbf{V}}_j = 0$ by the BD procedure. Therefore,

$$\mathbb{E}\left[\widetilde{\mathbf{H}}_k\widehat{\mathbf{V}}_j\widehat{\mathbf{V}}_j^{\text{H}}\widetilde{\mathbf{H}}_k^{\text{H}}\right] = \mathbb{E}\left[\mathbf{Z}_k\mathbf{S}_k\widehat{\mathbf{V}}_j\widehat{\mathbf{V}}_j^{\text{H}}\mathbf{S}_k^{\text{H}}\mathbf{Z}_k^{\text{H}}\right]. \quad (11)$$

In the extreme case, where the channels of all the K users are highly correlated, the inter-user interference in (11) can reach its maximum. In this case, the K users share same channel characteristics and clusters around the massive BS, i.e., $P_1 = P_2 = \cdots = P_K = P$ and $\mathbf{A}_1 = \mathbf{A}_2 = \cdots = \mathbf{A}_K = \mathbf{A}$. Therefore, the subscript k of P_k and $\mathbf{A}_k$ will be omitted in that proof. From our system model and feedback scheme in Sec. II, both the row-space of the quantized channel matrix, $\widehat{\mathbf{H}}_k$, and the spatial direction, $\widetilde{\mathbf{H}}_k$, lie in the row-space of $\mathbf{A}$. Since the row-space of $\widetilde{\mathbf{H}}_k$ can be orthogonally decomposed along the row-spaces of $\widehat{\mathbf{H}}_k$ and $\mathbf{S}_k$ as in (9), $\mathbf{S}_k$ must also lie in the row-space of $\mathbf{A}$. Thus, $\mathbf{S}_k$ can be written as $\mathbf{S}_k = \frac{1}{\sqrt{M}}\mathbf{T}_k\mathbf{A}$, where the rows of $\mathbf{T}_k \in \mathbb{C}^{N\times P}$ are orthonormal. This is valid since the row vectors of $\mathbf{A}$ are nearly orthogonal as M goes large in massive MIMO systems. Since the row-space of the quantized feedback channel matrix $\widehat{\mathbf{H}}_k$ is distributed in the row-space of $\mathbf{A}$, and from the BD procedure, the column-space of the precoder, $\widehat{\mathbf{V}}_j$, lies in the column-space of $\mathbf{A}^{\text{H}}$. Therefore, the precoding matrix, $\widehat{\mathbf{V}}_j$, can be expressed as $\widehat{\mathbf{V}}_j = \frac{1}{\sqrt{M}}\mathbf{A}^{\text{H}}\mathbf{Y}_j$, where $\mathbf{Y}_j \in \mathbb{C}^{P\times m}$ is a unitary matrix whose columns are orthonormal. Hence, substituting in (11), the interference term can be given as

$$\begin{aligned}\mathbb{E}\left[\widetilde{\mathbf{H}}_k\widehat{\mathbf{V}}_j\widehat{\mathbf{V}}_j^{\text{H}}\widetilde{\mathbf{H}}_k^{\text{H}}\right] &= \mathbb{E}\left[\mathbf{Z}_k\mathbf{T}_k\frac{\mathbf{A}\mathbf{A}^{\text{H}}}{M}\mathbf{Y}_j\mathbf{Y}_j^{\text{H}}\frac{\mathbf{A}\mathbf{A}^{\text{H}}}{M}\mathbf{T}_k^{\text{H}}\mathbf{Z}_k^{\text{H}}\right] \\ &= \mathbb{E}\left[\mathbf{Z}_k\mathbf{T}_k\mathbf{Y}_j\mathbf{Y}_j^{\text{H}}\mathbf{T}_k^{\text{H}}\mathbf{Z}_k^{\text{H}}\right] \end{aligned} \quad (12)$$

where the second equality holds as $\mathbf{A}\mathbf{A}^{\text{H}} = M\mathbf{I}_P$ when M goes large.

Lemma 1. *In the extreme case, $P_k = P$ and $\boldsymbol{A}_k = \boldsymbol{A}, \forall k$, when all users' channels are highly correlated, we have* $\mathbb{E}\left[\boldsymbol{Z}_k\boldsymbol{T}_k\boldsymbol{Y}_j\boldsymbol{Y}_j^H\boldsymbol{T}_k^H\boldsymbol{Z}_k^H\right] = \frac{N}{P-N}\mathbb{E}\left[\boldsymbol{Z}_k\boldsymbol{Z}_k^H\right] = \frac{N}{P-N}\frac{D}{N}$.

Proof. The subspace quantization of the channel matrix can be expressed as $\widehat{\mathbf{H}}_k = \frac{1}{\sqrt{M}}\widehat{\mathbf{G}}_k\mathbf{A}$, where $\widehat{\mathbf{G}}_k \in \mathbb{C}^{N\times P}$ is a matrix whose rows are orthonormal and its row-space represents the subspace quantization of $\mathbf{G}_k$, which is fed back by the users to the BS. Knowing that the row-space of $\mathbf{S}_k$ lies in the right null space of $\widehat{\mathbf{H}}_k$ as shown in (9), and considering that $\mathbf{S}_k = \frac{1}{\sqrt{M}}\mathbf{T}_k\mathbf{A}$, we have

$$\widehat{\mathbf{H}}_k{\mathbf{S}_k}^{\text{H}} = \frac{1}{M}\widehat{\mathbf{G}}_k\mathbf{A}\mathbf{A}^{\text{H}}{\mathbf{T}_k}^{\text{H}} = \widehat{\mathbf{G}}_k{\mathbf{T}_k}^{\text{H}} = 0. \quad (13)$$

Therefore, the row-space of $\mathbf{T}_k$ is distributed in the right null space of $\widehat{\mathbf{G}}_k$. On the other hand, as previously mentioned, the BD precoding matrix can be expressed as $\widehat{\mathbf{V}}_j = \frac{1}{\sqrt{M}}\mathbf{A}^{\text{H}}\mathbf{Y}_j$. Since the column-space of the BD precoding matrix $\widehat{\mathbf{V}}_j$ is orthogonal to the row-space of $\widehat{\mathbf{H}}_k$, i.e.,

$$\widehat{\mathbf{H}}_k\widehat{\mathbf{V}}_j = \frac{1}{M}\widehat{\mathbf{G}}_k\mathbf{A}\mathbf{A}^{\text{H}}\mathbf{Y}_j = \widehat{\mathbf{G}}_k\mathbf{Y}_j = 0. \quad (14)$$

Therefore, the column-space of $\mathbf{Y}_j$ is isotropically distributed in the right null space of $\widehat{\mathbf{G}}_k$. Now we have proved that both the row-space of $\mathbf{T}_k$ and the column-space of $\mathbf{Y}_j$ are isotropic sub-spaces in the null space of $\widehat{\mathbf{G}}_k$. Based on [13], we have

$$\mathbb{E}\left[\mathbf{Z}_k\mathbf{T}_k\mathbf{Y}_j\mathbf{Y}_j^{\mathrm{H}}\mathbf{T}_k^{\mathrm{H}}\mathbf{Z}_k^{\mathrm{H}}\right] = \frac{N}{P-N}\mathbb{E}\left[\mathbf{Z}_k\mathbf{Z}_k^{\mathrm{H}}\right] = \frac{N}{P-N}\frac{D}{N}\mathbf{I}_N, \tag{15}$$

□

where D is the average subspace quantization error which is defined as

$$D = \mathbb{E}\left[d^2(\widetilde{\mathbf{H}}_k, \widehat{\mathbf{H}}_k)\right], \tag{16}$$

where $d(.,.)$ is the chordal distance which is defined as [14]

$$d(\widetilde{\mathbf{H}}_k, \widehat{\mathbf{H}}_k) = \left[N_k - \left\|\widetilde{\mathbf{H}}_k\widehat{\mathbf{H}}_k^{\mathrm{H}}\right\|_{\mathrm{F}}^2\right]^{1/2}. \tag{17}$$

Hence, the interference term is given as

$$\mathbb{E}\left[\widetilde{\mathbf{H}}_k\widehat{\mathbf{V}}_j\widehat{\mathbf{V}}_j^{\mathrm{H}}\widetilde{\mathbf{H}}_k^{\mathrm{H}}\right] = \frac{D}{P-N}\mathbf{I}_N, \tag{18}$$

and the rate gap can be upper bounded using the following equation

$$\Delta R_{\text{QUANT}} \leq N\log_2\left(1 + \frac{\gamma(K-1)MP}{K(P-N)}D\right). \tag{19}$$

B. Quantization Error

In this subsection, the quantization error, D, of the spatial direction of the kth user, is calculated when the AoD based quantized channel feedback is used. The spatial direction of the kth user, $\widetilde{\mathbf{H}}_k$, can be written as $\widetilde{\mathbf{H}}_k = \frac{1}{\sqrt{M}}\widetilde{\mathbf{G}}_k\mathbf{A}_k$, where $\widetilde{\mathbf{G}}_k \in \mathbb{C}^{N\times P}$ is a matrix whose rows are orthonormal and its row-space is the row-space of $\mathbf{G}_k$. Hence, the quantization error can be given as

$$D = \mathbb{E}\left[N - \left\|\widetilde{\mathbf{H}}_k\widehat{\mathbf{H}}_k^{\mathrm{H}}\right\|_{\mathrm{F}}^2\right] = \mathbb{E}\left[N - \left\|\frac{1}{M}\widetilde{\mathbf{G}}_k\mathbf{A}_k\mathbf{A}_k^{\mathrm{H}}\widehat{\mathbf{G}}_k^{\mathrm{H}}\right\|_{\mathrm{F}}^2\right] \tag{20}$$

$$\overset{(a)}{\approx} \mathbb{E}\left[N - \left\|\widetilde{\mathbf{G}}_k\widehat{\mathbf{G}}_k^{\mathrm{H}}\right\|_{\mathrm{F}}^2\right]. \tag{21}$$

where $\widehat{\mathbf{G}}_k$ represents the subspace quantization of the row-space of $\widetilde{\mathbf{G}}_k$ and $\mathbf{A}_k\mathbf{A}_k^{\mathrm{H}} \approx M\mathbf{I}_P$. Both $\widetilde{\mathbf{G}}_k$ and $\widehat{\mathbf{G}}_k$ are isotropically distributed subspaces on the P-dimensional space. Then, we can bound the quantization error as [13]

$$D \leq \bar{D} = \frac{\Gamma(\frac{1}{T})}{T}(C_{PN})^{-\frac{1}{T}}2^{-\frac{B}{T}}, \tag{22}$$

where $T = N(P-N)$ and $C_{PN} = \frac{1}{T!}\prod_{i=1}^{N}\frac{(P-i)!}{(N-i)!}$. The parameter B is the number of feedback bits used to quantize the row-space of $\mathbf{G}_k$, where 2^B is the subspace codebook size.

IV. Quantized vs. Analog Channel Feedback

In this section, we firstly discuss an analog channel feedback technique based on the channel subspace $\mathbf{A}$. Then, this channel subspace-based analog feedback technique is analyzed by evaluating the rate gap between the ideal case of the perfect CSI and the practical case of the analog feedback technique. Finally, the proposed quantized feedback technique using the BD-based subspace codebook is compared with the analog feedback technique.

A. Analog channel feedback technique

In the traditional analog channel feedback technique, each user k feeds back the elements of its channel matrix $\mathbf{H}_k$ explicitly without quantization through the uplink noisy channel [15]. In this paper, since $\mathbf{H}_k = \mathbf{G}_k\mathbf{A}_k$, only the elements of the $N_k \times P_k$ path gain matrix $\mathbf{G}_k$ are fed back through a noisy uplink channel, as long as we assume that path AoDs, i.e., the steering matrix, $\mathbf{A}_k$, is known at the BS. The observation of the path gain matrix $\mathbf{G}_k$ at the BS after being transmitted over the noisy uplink channel is given by

$$\mathbf{G}_{Ana,k} = \sqrt{\mu\gamma_U}\mathbf{G}_k + \mathbf{N}_{U,k}, \tag{23}$$

where γ_U is the uplink SNR and $\mathbf{N}_{U,k}$ is the uplink complex Gaussian noise, where each element has zero mean and unit variance. The scale factor μ denotes the number of channel uses to feedback one element of the path gain matrix $\mathbf{G}_k$. The MMSE estimate of the path gain matrix at the BS is given as

$$\breve{\mathbf{G}}_k = \frac{\sqrt{\mu\gamma_U}}{1+\mu\gamma_U}\mathbf{G}_{Ana,k}. \tag{24}$$

It is convenient to express $\mathbf{G}_k$ in terms of the estimate $\breve{\mathbf{G}}_k$ and estimation noise as follows

$$\mathbf{G}_k = \breve{\mathbf{G}}_k + \mathbf{E}_{\mathbf{G}_k}, \tag{25}$$

where $\breve{\mathbf{G}}_k$ and $\mathbf{E}_{\mathbf{G}_k}$ are mutually independent and their elements are zero mean Gaussian random variables with variances $\mu\gamma_U\sigma^2_{\mathbf{E}_{\mathbf{G}_k}}$ and $\sigma^2_{\mathbf{E}_{\mathbf{G}_k}} = (1+\mu\gamma_U)^{-1}$ respectively. Then, by utilizing the path AoD information, i.e., steering matrix $\mathbf{A}_k$, the BS can recover the channel matrix $\breve{\mathbf{H}}_k$ obtained from the proposed channel subspace based analog feedback as

$$\breve{\mathbf{H}}_k = \breve{\mathbf{G}}_k\mathbf{A}_k. \tag{26}$$

Hence, the channel matrix can be rewritten as

$$\mathbf{H}_k = \mathbf{G}_k\mathbf{A}_k = \breve{\mathbf{H}}_k + \mathbf{E}_{\mathbf{G}_k}\mathbf{A}_k \tag{27}$$

B. Rate gap of the analog channel feedback

Like the case of quantized channel feedback, BD is considered as the zero forcing precoding technique for multi-user downlink transmission. This can be realized based on the channel matrices of the users, $\breve{\mathbf{H}}_k$, that were fed back to the massive BS using the analog feedback technique. Hence, the BS can compute the unitary precoding matrices, $\breve{\mathbf{V}}_k$, following the normal BD procedure. After the downlink transmission, the received vector at user k can be written as

$$\mathbf{y}_k = \mathbf{H}_k\breve{\mathbf{V}}_k\mathbf{m}_k + \mathbf{H}_k\sum_{j=1,j\neq k}^{K}\breve{\mathbf{V}}_j\mathbf{m}_j + \mathbf{n}_k \tag{28}$$

Following appendix C in [13], the rate gap resulting from analog channel feedback, $\Delta R_{\text{Analog}} = R_{\text{Ideal}} - R_{\text{Analog}}$, is bounded as

$$\Delta R_{\text{Analog}} \leq \mathbb{E}\log_2\left|\mathbf{I}_N + \frac{\gamma}{KN}\sum_{j=1,j\neq k}^{K}\breve{\mathbf{V}}_j^{\mathrm{H}}\mathbf{H}_k^{\mathrm{H}}\mathbf{H}_k\breve{\mathbf{V}}_j\right|. \tag{29}$$

By applying Jensens inequality to the above equation and by substituting with (27) noting that $\check{\mathbf{H}}_k\check{\mathbf{V}}_j = 0$ for $k \neq j$, the analog rate gap can be bounded as

$$\Delta R_{\text{Analog}} \leq \log_2 \left| \mathbf{I}_N + \frac{\gamma(K-1)}{KN} \mathbb{E}\left[\check{\mathbf{V}}_j^{\text{H}} \mathbf{A}_k^{\text{H}} \mathbf{E}_{\mathbf{G}_k}^{\text{H}} \mathbf{E}_{\mathbf{G}_k} \mathbf{A}_k \check{\mathbf{V}}_j \right] \right|. \tag{30}$$

Now, an upper bound on the interference from other users to the kth user, $\mathbb{E}\left[\check{\mathbf{V}}_j^{\text{H}} \mathbf{A}_k^{\text{H}} \mathbf{E}_{\mathbf{G}_k}^{\text{H}} \mathbf{E}_{\mathbf{G}_k} \mathbf{A}_k \check{\mathbf{V}}_j \right]$, is calculated. Clearly, an upper bound of this expectation can be reached in an extreme case, where all the K users have strongly correlated channel conditions. In this case, the channel subspace, $\mathbf{A}_k$, of all users will be the same, i.e., $P_1 = P_2 = \cdots = P_K = P$ and $\mathbf{A}_1 = \mathbf{A}_2 = \cdots = \mathbf{A}_K = \mathbf{A}$. Consequently, the subscript k is omitted of P_k and $\mathbf{A}_k$ throughout the rest of this section. Following the BD procedure, and as discussed in Sec. III-A, the precoding matrix $\check{\mathbf{V}}_j$ can be expressed as $\check{\mathbf{V}}_j = \frac{1}{\sqrt{M}} \mathbf{A}^{\text{H}} \mathbf{U}_j$, where $\mathbf{U}_j \in \mathbb{C}^{P \times N}$ is a unitary matrix whose columns are orthonormal. Hence, inter-user interference can be bounded as

$$\begin{aligned}\mathbb{E}\left[\check{\mathbf{V}}_j^{\text{H}} \mathbf{A}^{\text{H}} \mathbf{E}_{\mathbf{G}_k}^{\text{H}} \mathbf{E}_{\mathbf{G}_k} \mathbf{A} \check{\mathbf{V}}_j \right] &\leq \frac{1}{M} \mathbb{E}\left[\mathbf{U}_j^{\text{H}} \mathbf{A}\mathbf{A}^{\text{H}} \mathbf{E}_{\mathbf{G}_k}^{\text{H}} \mathbf{E}_{\mathbf{G}_k} \mathbf{A}\mathbf{A}^{\text{H}} \mathbf{U}_j \right] \\ &= M \mathbb{E}\left[\mathbf{U}_j^{\text{H}} \mathbb{E}\left[\mathbf{E}_{\mathbf{G}_k}^{\text{H}} \mathbf{E}_{\mathbf{G}_k} \right] \mathbf{U}_j \right], \end{aligned} \tag{31}$$

knowing that $\mathbf{A}_k \mathbf{A}_k^{\text{H}} \approx M \mathbf{I}_P$. By substituting $\mathbb{E}\left[\mathbf{E}_{\mathbf{G}_k}^{\text{H}} \mathbf{E}_{\mathbf{G}_k} \right] = N \sigma_{\mathbf{E}_{\mathbf{G}_k}}^2 \mathbf{I}_P$, we get an upper bound on the interference as

$$\mathbb{E}\left[\check{\mathbf{V}}_j^{\text{H}} \mathbf{A}^{\text{H}} \mathbf{E}_{\mathbf{G}_k}^{\text{H}} \mathbf{E}_{\mathbf{G}_k} \mathbf{A} \check{\mathbf{V}}_j \right] \leq M N \sigma_{\mathbf{E}_{\mathbf{G}_k}}^2 \mathbf{I}_N. \tag{32}$$

By substituting (32) in (30), we can express the upper bound of the rate gap of the analog feedback scheme as

$$\begin{aligned}\Delta R_{\text{Analog}} &\leq \log_2 \left| \mathbf{I}_N + \frac{\gamma(K-1)}{K} M \sigma_{\mathbf{E}_{\mathbf{G}_k}}^2 \mathbf{I}_N \right| \\ &= N \log_2 \left(1 + \frac{\gamma(K-1)}{K} M (1 + \mu \gamma_U)^{-1} \right). \end{aligned} \tag{33}$$

C. Comparison between quantized and analog channel feedback

Recall the rate gap of the quantized channel feedback in (19). By substituting (22) in (19), the rate gap of the quantized channel feedback can be further simplified as

$$\Delta R_{\text{QUANT}} \leq N \log_2 \left(1 + C 2^{-\frac{B}{T}} \right), \tag{34}$$

where C is a constant and $T = N(P-N)$. We assume that the link used to feedback the quantized channel is error-free at its capacity [16], i.e., we can feedback $\log_2(1+\gamma_U)$ bits with no errors per one channel use. For fair comparison, we need to equate the allocated feedback resources for both the analog and digital feedback schemes. In the analog feedback, the path gains matrix $\mathbf{G}_k$ is transmitted element by element requiring a number of $\mu P N$ channel uses, where μ is the number of channel uses per one element. Consequently, the quantized channel feedback scheme can transmit $B = \mu P N \log_2(1+\gamma_U)$ bits using the same feedback resource as the analog channel feedback. By substituting $B = \mu P N \log_2(1+\gamma_U)$ into (34), the rate gap of the quantized feedback scheme can be rewritten as

$$\Delta R_{\text{QUANT}} \leq N \log_2 \left(1 + C (1+\gamma_U)^{-\frac{\mu P N}{T}} \right). \tag{35}$$

Now, the rate gaps of both the analog and quantized channel feedback schemes are compared as the scaling parameter μ grows large for a constant uplink SNR, γ_U. In the analog channel feedback scheme (33), the value $2^{\Delta R_{\text{Analog}}}$ decays inversely with the scale factor μ as μ increases. On the other hand, in the quantized channel feedback scheme (35), the value $2^{\Delta R_{\text{Analog}}}$ decays exponentially with the scaling factor μ. Thus, we can easily see that the quantized channel feedback scheme outperforms the analog one at large values of the scale factor μ.

V. Water-filling based analog channel feedback

In this section, we discuss making use of the additional information given by the analog channel feedback to achieve higher performance than the quantized one in the low SNR region. In analog channel feedback, the entries of the path gains' matrix, $\mathbf{G}_k(i,j)$, are individually fed back to the massive BS. Therefore, the BS can calculate the singular values of the effective downlink channel of user k, and hence performing power allocation among its streams. This information is not available in the quantized feedback scheme as we only feedback a unitary matrix that represents the row-space of $\mathbf{G}_k$, not the actual matrix. Hence, the performance of analog channel feedback scheme can be further improved at low downlink SNR region due to optimal power allocation (water-filling) across data streams. At low SNR region, the system noise is more dominant than the quantization noise of the channels. Hence, at low downlink SNR region, performing power allocation across data streams may be more beneficial than using the quantized feedback scheme which provides a better approximation of the actual CSI.

The performance of BD based precoding degrades in the low SNR region as the zero-forcing techniques generally suffer when the noise level is high. Consequently, in this section, we aim to raise the performance of our system in the low SNR region by optimally allocating power across the multiple data streams. This optimal power allocation aims to maximize the system's total sum rate. We aim to calculate the power allocation diagonal matrices, $\mathbf{\Delta_k}$, of the users whose diagonal elements represent the power fractions that maximize the whole system throughput. Thus, the modified precoding matrix for the kth user at the massive BS becomes $\mathbf{V}_k \mathbf{\Delta_k^{1/2}}$. Let's assume at first the the BS has perfect CSI information, then the inter-user interference is totally cancelled When BD precoding is considered at the BS. In this case, the sum rate is given as

$$R_{\text{tot}} = \mathbb{E}\left\{ \sum_{k=1}^{K} \log_2 \left| \mathbf{I}_{N_k} + \mathbf{H}_k \mathbf{V}_k \mathbf{\Delta_k} \mathbf{V}_{\mathbf{k}}^{\text{H}} \mathbf{H}_{\mathbf{k}}^{\text{H}} \right| \right\}, \tag{36}$$

where the diagonal elements of $\mathbf{\Delta_k}$ scale the power transmitted into each of the columns of $\mathbf{V}_k$. As long as BD precoding forces the interference of other users to be zero, each user in the system is seen as a point-to-point MIMO link by the massive BS. Hence, the system's sum rate rate can be given as [11]

$$R_{\text{tot}} = \mathbb{E}\left\{ \sum_{k=1}^{K} \log_2 \left| \mathbf{I}_{N_k} + \mathbf{\Lambda}_{\mathbf{k}}^{\mathbf{2}} \mathbf{\Delta}_{\mathbf{k}} \right| \right\}, \tag{37}$$

where the diagonal elements of the diagonal matrix $\mathbf{\Lambda}_{\mathbf{k}}$, $\sigma_{k,i}$, represents the singular values of the kth user's effective channel, $\mathbf{H}_k \mathbf{V}_k$. Now, the power allocation problem can be expressed as

$$\begin{aligned} \max_{\delta_{k,i}} \quad & \sum_{k=1}^{K} \sum_{i=1}^{N_k} \log_2 \left(1 + \sigma_{k,i}^2 \delta_{k,i}\right) \\ \text{s.t.} \quad & \sum_{k=1}^{K} \sum_{i=1}^{N_k} \delta_{k,i} \leq \gamma, \end{aligned}$$

where γ is the total transmitted power at the massive BS. Clearly, the above optimization problem is convex and it has a well known closed form solution [17] when solved using the Lagrange multiplier method. The closed form solution of the above problem is

$$\delta_{k,i}^{*} = \left(\frac{1}{\alpha} - \frac{1}{\sigma_{k,i}^2} \right)^{+}, \tag{38}$$

where $x^{+} = x$ when $x \geq 0$ and equals to zero elsewhere, and α is calculated such that the total transmitted power at the massive BS is equal to γ. Therefore, α is the solution of the following equation

$$\sum_{k=1}^{K} \sum_{i=1}^{N_k} \left(\frac{1}{\alpha} - \frac{1}{\sigma_{k,i}^2} \right)^{+} = \gamma. \tag{39}$$

Now, when the analog channel feedback is used in practical systems, the elements of the path gains matrix, $\mathbf{G}_k$, is fed back to the BS so that the BS can have $\breve{\mathbf{G}}_k$. Then, the BS uses $\breve{\mathbf{G}}_k$ to generate the estimated channel matrices, $\breve{\mathbf{H}}_k$, and use them in the calculation of the precoding matrices, $\breve{\mathbf{V}}_k$, using the BD procedure. Therefore, power allocation across the multiple data streams is adopted at the massive BS as discussed earlier in this section. The BS is this case computes the power allocation diagonal matrix $\breve{\mathbf{\Delta}}_k$ based on the singular values, $\breve{\mathbf{\Lambda}}_k$, of the effective channel, $\breve{\mathbf{H}}_k \breve{\mathbf{V}}_k$, of user k. Hence, the per-user rate considering analog channel feedback and water filling across the data streams is written as

$$\begin{aligned} R_{\text{WF},k}(\mathbf{\Delta}_{\mathbf{k}}) &= \mathbb{E} \log_2 \left| \mathbf{I}_{N_k} + \sum_{j=1}^{K} \mathbf{H}_k \breve{\mathbf{V}}_j \breve{\mathbf{\Delta}}_j \breve{\mathbf{V}}_j^{\mathrm{H}} \mathbf{H}_k^{\mathrm{H}} \right| - \\ & \quad \mathbb{E} \log_2 \left| \mathbf{I}_{N_k} + \sum_{j=1, j \neq k}^{K} \mathbf{H}_k \breve{\mathbf{V}}_j \breve{\mathbf{\Delta}}_j \breve{\mathbf{V}}_j^{\mathrm{H}} \mathbf{H}_k^{\mathrm{H}} \right|. \end{aligned} \tag{40}$$

VI. Simulation Results

In this section, a simulation study is carried out to verify the conclusions of this paper. The system parameters are set as follows. The number of antennas at the BS is $M = 128$, the number of users in the system is $K = 8$, the number of antennas at each user is equal to the number of parallel data streams transmitted to the user, i.e., $N_k = N = 2$. The number of resolvable paths from the BS to the users is $P_k = P = 4$. The path AoDs of the users are independent and uniformly distributed in $\left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$. The per-user rates for the cases of ideal CSI, practical channel feedback, and water-filling based analog channel feedback are computed according to (6), (7), and (40), respectively. In the case of quantized channel feedback, the subspace quantization is based on random subspace codebook framework.

Fig. 1 compares the rate gap of the quantized feedback scheme against the analog feedback scheme with and without power allocation across the data streams. The number

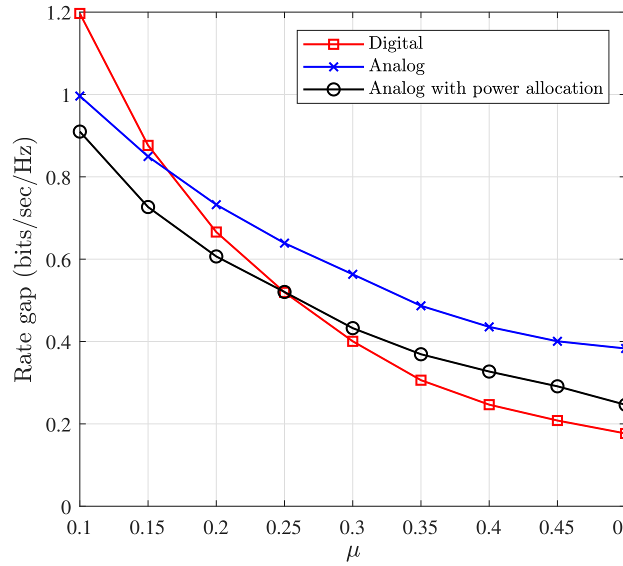

Figure 1: Rate gap comparison between quantized and analog channel feedback with downlink SNR= 10dB and $\gamma_U = 5$

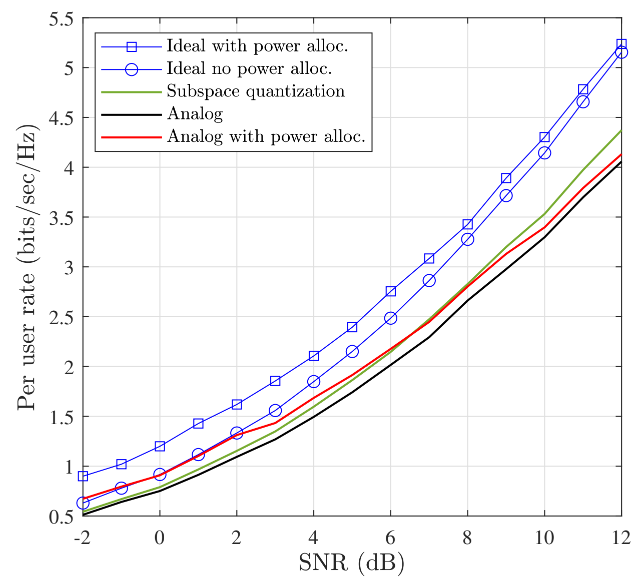

Figure 2: Per-user rate comparison under different schemes vs. downlink SNR with $\mu = 0.3$ and $\gamma_U = 5$

of feedback bits in case of quantized feedback is set as $B = \mu PN \log_2(1 + \gamma_U)$. As shown in Fig. 1, and for a constant uplink SNR of $\gamma_U = 5$, the rate gap of the quantized feedback scheme decays exponentially, which is faster than that of the analog feedback scheme, which decays inversely as μ increases. This result is consistent with our conclusion in Sec. IV-C. Fig. 1 also shows the impact of using power allocation across the data streams on the rate gap of the analog feedback scheme.

Fig. 2 compares the per-user rate of the ideal CSI, quantized feedback and analog feedback schemes when $\mu = 0.3$ and a constant uplink SNR $\gamma_U = 5$. The graph shows that, under these parameters, the subspace quantization feedback scheme outperforms the analog feedback scheme over the whole SNR range. However, when performing optimal power allocation across the data streams, the analog feedback scheme outperforms the quantized scheme up to an SNR of = 6dB. Beyond this point, the quantized feedback scheme achieves better performance again as it provides a better approximation of the actual CSI and the system noise becomes less significant.

VII. Conclusions

In this paper, we have quantified and compared two CSI feedback schemes' performance, namely, the quantized and the analog channel feedback schemes for AoD based massive MIMO systems with multiple antenna users. We derive rate gap bounds for the two schemes in comparison to the system with CSI feedback. The derived rate gap bounds of both schemes show that the quantized feedback scheme's rate gap decays exponentially, while the rate gap of the analog feedback scheme decays inversely as μ increases. Our results also show that power allocation through water-filling can significantly enhance the system performance. We have shown that the analog feedback scheme with water-filling outperforms the quantized feedback scheme at low SNRs. However, at higher SNRs, the quantized feedback scheme can achieve better performance.

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