Response under a general periodic force

If the forcing function is periodic, we can use the Fourier series and the principle of superposition to get the response. The Fourier series states that a periodic function can be represented as a series of sines and cosines:

$$F(t) = \frac{a_0}{2} + a_1 \cos \omega t + a_2 \cos 2\omega t + ... + b_1 \sin \omega t + b_2 \sin 2\omega t + ...$$
$$= \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos j\omega t + b_j \sin j\omega t$$
$$a_j = \frac{2}{T}\int_0^T F(t)\cos(j\omega t)\,dt \quad , \quad j = 0,1,2,\dots$$
$$b_j = \frac{2}{T}\int_0^T F(t)\sin(j\omega t)\,dt \quad , \quad j = 1,2,3,\dots$$

where $T = 2\pi/\omega$ is the period. Now the equation of motion can be written as:

$$m\ddot{x} + c\dot{x} + kx = F(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos j\omega t + b_j \sin j\omega t$$

Using the principle of superposition, the steady-sate solution of this equation is the sum of the steady-state solutions of:

$$m\ddot{x} + c\dot{x} + kx = \frac{a_0}{2}$$
$$m\ddot{x} + c\dot{x} + kx = a_j \cos j\omega t$$
$$m\ddot{x} + c\dot{x} + kx = b_j \sin j\omega t$$

The particular solution of the 1st equation is:

$$x_p(t) = \frac{a_0}{2k}$$

The particular solutions of the 2nd and 3rd equations are:

$$x_p(t) = \frac{a_j/k}{\sqrt{\left(1 - j^2 r^2\right)^2 + \left(2\zeta jr\right)^2}} \cos\left(j\omega t - \phi_j\right)$$

$$x_p(t) = \frac{b_j/k}{\sqrt{\left(1 - j^2 r^2\right)^2 + \left(2\zeta jr\right)^2}} \sin\left(j\omega t - \phi_j\right)$$

$$\phi_j = \tan^{-1}\left(\frac{2\zeta jr}{1 - j^2 r^2}\right)$$

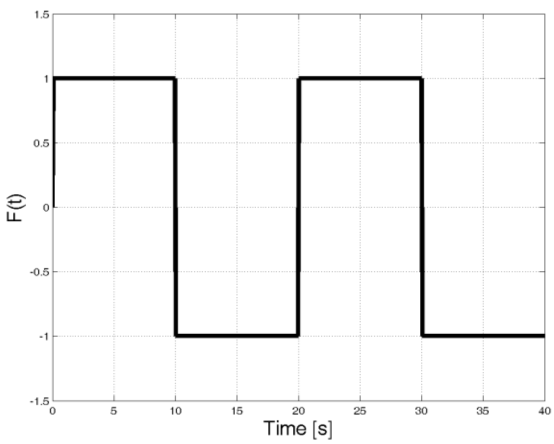
$$r = \frac{\omega}{\omega_n}$$

Then add up all the sums to get the complete steady-state solution as:

$$x_p(t) = \frac{a_o}{2k} + \sum_{j=1}^{\infty} \frac{a_j/k}{\sqrt{\left(1 - j^2 r^2\right)^2 + \left(2\zeta jr\right)^2}} \cos\left(j\omega t - \phi_j\right) + \sum_{j=1}^{\infty} \frac{b_j/k}{\sqrt{\left(1 - j^2 r^2\right)^2 + \left(2\zeta jr\right)^2}} \sin\left(j\omega t - \phi_j\right)$$

Observe that if $j\omega = \omega_n$, the amplitude will be significantly large, especially for small j and ζ. Further, as j becomes large, the amplitude becomes smaller and the corresponding terms tend to zero. How many terms do you need to include?

Example: Obtain the steady-state response of a dynamic system having m=1, c=0.7 and k=1 when subjected to the force shown.



Solution: here we have:

$$T = 20 \quad , \quad \omega = 2\pi/T = 2\pi/20 = \pi/10$$

and the forcing function is given by:

$$F(t) = 1 \qquad 0 \le t \le 10$$

$$F(t) = -1 \quad 10 \le t \le 20$$

The Fourier series of the forcing function is given by:

$$F\left(t\right) = \frac{a_0}{2} + a_1 \cos \omega t + a_2 \cos 2\omega t + ... + b_1 \sin \omega t + b_2 \sin 2\omega t + ...$$

To get the constants, we have:

$$a_0 = \frac{2}{T}\int_0^T F\left(t\right)dt = \frac{2}{20}\left[\int_0^{10} dt - \int_{10}^{20} dt\right] = 0.1\left[10 - \left(20 - 10\right)\right] = 0$$

$$\begin{aligned} a_j &= \frac{2}{T}\int_0^T F\left(t\right)\cos\left(j\omega t\right)dt \\ &= \frac{2}{20}\left[\int_0^{10} \cos\left(j\frac{\pi}{10}t\right)dt - \int_{10}^{20} \cos\left(j\frac{\pi}{10}t\right)dt\right] \\ &= 0.1\left[\frac{10}{j\pi}\sin j\frac{\pi}{10}t\Big|_0^{10} - \frac{10}{j\pi}\sin j\frac{\pi}{10}t\Big|_{10}^{20}\right] = 0 \end{aligned}$$

i.e. all cosine terms vanish

$$\begin{aligned} b_j &= \frac{2}{T}\int_0^T F\left(t\right)\sin\left(j\omega t\right)dt \\ &= \frac{2}{20}\left[\int_0^{10} \sin\left(j\frac{\pi}{10}t\right)dt - \int_{10}^{20} \sin\left(j\frac{\pi}{10}t\right)dt\right] \\ &= 0.1\left[\frac{-10}{j\pi}\cos j\frac{\pi}{10}t\Big|_0^{10} - \frac{-10}{j\pi}\cos j\frac{\pi}{10}t\Big|_{10}^{20}\right] \\ &= 0.1\left[\frac{-10}{j\pi}\left(\cos j\pi - 1\right) + \frac{10}{j\pi}\left(\cos 2j\pi - \cos j\pi\right)\right] \end{aligned}$$

If j is odd,

$$b_j = 0.1\left[\frac{-10}{j\pi}(-2)+\frac{10}{j\pi}(2)\right]=\frac{4}{j\pi}$$

If j is even,

$$b_j = 0.1\left[\frac{-10}{j\pi}(0)+\frac{10}{j\pi}(0)\right]=0$$

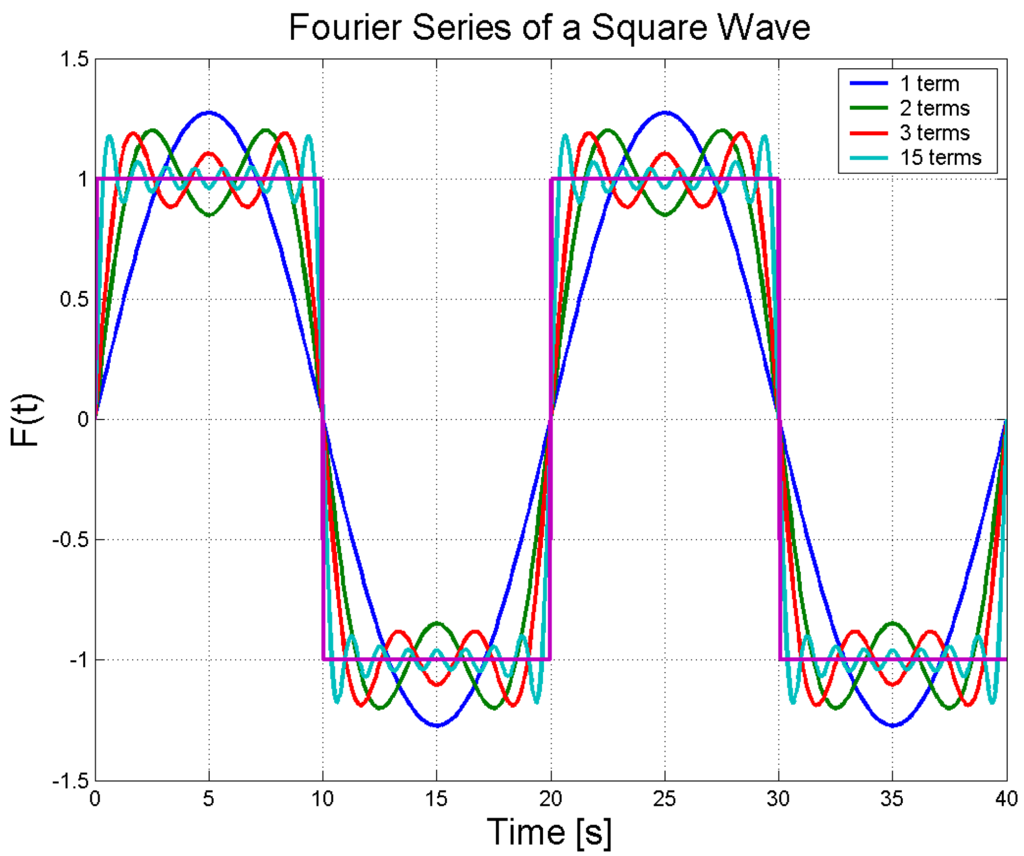
i.e. all even terms vanish.

In this way, the force can be represented by a Fourier series as:

$$F(t)=b_1\sin\omega_1 t+b_3\sin\omega_3 t+b_5\sin\omega_5 t+\cdots=\sum_{j=1}^{\infty} b_j\sin(j\omega t)\quad,\quad j=1,3,5,7,\ldots$$

$$=\frac{4}{\pi}\sin\frac{\pi}{10}t+\frac{4}{3\pi}\sin\frac{3\pi}{10}t+\frac{4}{5\pi}\sin\frac{5\pi}{10}t+\cdots=\sum_{j=1}^{\infty}\frac{4}{j\pi}\sin\left(j\frac{\pi}{10}t\right)\quad,\quad j=1,3,5,7,\ldots$$

or graphically as:



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%   *******************************************************
%   Periodic response of a dynamic system to a square input waveform
%   *******************************************************
clear; close all;
m=1; c=0.7; k=1; % Parameters

wn=sqrt(k/m); zi=c/wn/2; w=pi/10;

t=linspace(0,40,500); x=zeros(size(t)); r=w/wn;
for jj=1:2:19
    a(jj)=4/pi/jj;
    X(jj)=a(jj)/k/sqrt((1-jj^2*r^2)^2+(2*zi*jj*r)^2);   % term in summation
    phi(jj)=atan2(2*zi*jj*r,1-jj*jj*r*r);
    x(jj+2,:)=x(jj,:)+X(jj)*sin(w*jj*t-phi(jj));
end
u=sign(sin(w*t));

figure(1)
plot(t,x([3 5 7 21],:),t,sign(sin(w*t)),'linewidth',2);grid
legend('1 term','2 terms','3 terms','19 terms')
xlabel('Time [s]','fontsize',18);ylabel('x(t)','fontsize',18)
title('Response to Periodic Input','fontsize',18);

figure(2)
plot(1:2:19,X(1:2:19),'o','markersize',10,'linewidth',4);grid
ylabel('Magnitude of term in summation','fontsize',18)
xlabel('Summation term','fontsize',18)
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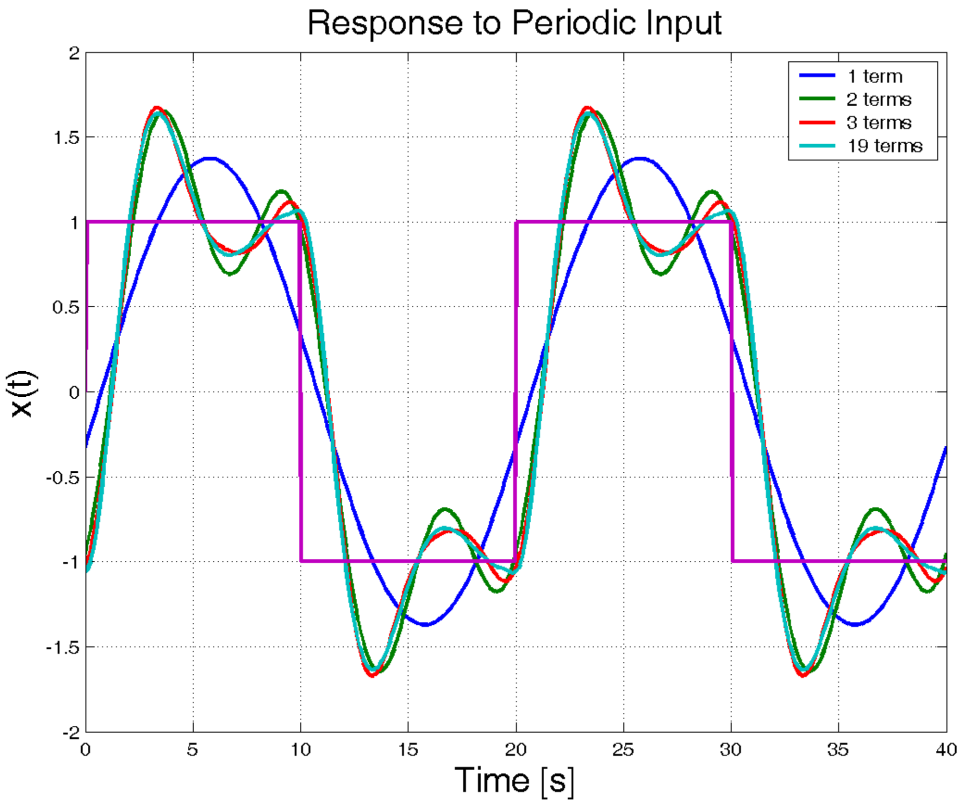
The system response is shown below:



The contribution of each term in the summation is determined from: