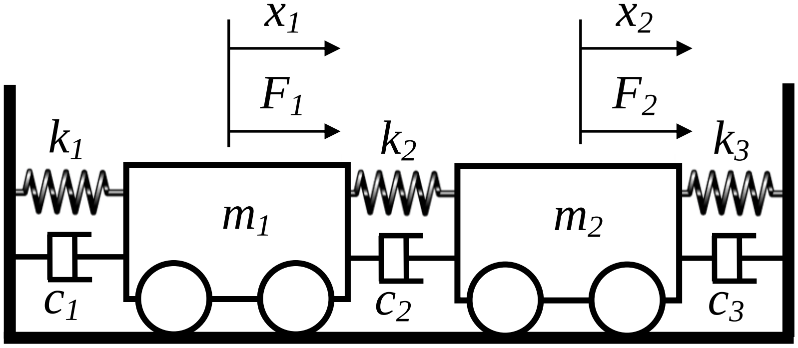
TWO DEGREE OF FREEDOM SYSTEMS

The number of degrees of freedom (DOF) of a system is the number of independent coordinates necessary to define motion. Also, the number of DOF is equal to the number of masses multiplied by the number of independent ways each mass can move. Consider the 2 DOF system shown below.



From Newton's law, the equations of motion are:

$$m_1\ddot{x}_1 = -k_1x_1 - c_1\dot{x}_1 + k_2\left(x_2 - x_1\right) + c_2\left(\dot{x}_2 - \dot{x}_1\right) + F_1$$
$$m_2\ddot{x}_2 = -k_3x_2 - c_3\dot{x}_2 - k_2\left(x_2 - x_1\right) - c_2\left(\dot{x}_2 - \dot{x}_1\right) + F_2 \quad (1)$$

Rearranging gives:

$$m_1\ddot{x}_1 + \left(c_1 + c_2\right)\dot{x}_1 + \left(k_1 + k_2\right)x_1 - c_2\dot{x}_2 - k_2x_2 = F_1$$
$$m_2\ddot{x}_2 + \left(c_2 + c_3\right)\dot{x}_2 + \left(k_2 + k_3\right)x_2 - c_2\dot{x}_1 - k_2x_1 = F_2 \quad (2)$$

These equations can be written in matrix form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \quad (3)$$

Defining:

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \quad \{f\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$[M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad [C] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \quad [K] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

we get:

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F\} \quad (4)$$

Free Undamped Vibration

Setting the damping $[C]$and forcing $\{F\}$terms to zero, we get:

$$m_1\ddot{x}_1 + \left(k_1 + k_2\right)x_1 - k_2x_2 = 0$$
$$m_2\ddot{x}_2 + \left(k_2 + k_3\right)x_2 - k_2x_1 = 0 \quad (5)$$

Solution: we can assume that each mass undergoes harmonic motion of the same frequency and phase. This is proved on page 4. The solution is thus written as:

$$x_1 = X_1 \cos\omega t$$
$$x_2 = X_2 \cos\omega t \quad (6)$$

or:

$$\{x\} = \{X\}\cos\omega t \tag{7}$$

where

$$\{X\} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \tag{8}$$

Substituting into the equation of motion yields:

$$-\omega^2[M]\{x\} + [K]\{x\} = \{0\} \Rightarrow \big[[K] - \omega^2[M]\big]\{x\} = \{0\} \tag{9}$$

This is an eigenvalue problem. For a non-trivial solution, the determinant must vanish so we have:

$$\big\|[K] - \omega^2[M]\big\| = 0 \tag{10}$$

Or

$$\begin{vmatrix} -m_1\omega^2 + k_1 + k_2 & -k_2 \\ -k_2 & -m_2\omega^2 + k_2 + k_3 \end{vmatrix} = 0 \tag{11}$$

For our problem, this results in:

$$\left(-m_1\omega^2 + k_1 + k_2\right)\left(-m_2\omega^2 + k_2 + k_3\right) - \left(-k_2\right)\left(-k_2\right) = 0 \tag{12}$$

combining terms we get:

$$m_1m_2\left(\omega^2\right)^2 + \left(-m_1\left(k_2 + k_3\right) - m_2\left(k_1 + k_2\right)\right)\omega^2 + \left(k_1k_2 + k_2k_3 + k_1k_3\right) = 0 \tag{13}$$

which is a quadratic equation in terms of ω^2. From this we can get:

$$\omega^2 = \frac{m_1\left(k_2 + k_3\right) + m_2\left(k_1 + k_2\right) \pm \sqrt{\left(m_1\left(k_2 + k_3\right) + m_2\left(k_1 + k_2\right)\right)^2 - 4m_1m_2\left(k_1k_2 + k_2k_3 + k_1k_3\right)}}{2m_1m_2} \tag{14}$$

These values of ω are the *natural frequencies* of the system. The values of X_1 and X_2 remain to be determined. To simplify the analysis, let $m_1=m_2=m$ and $k_1=k_2=k_3=k$. The determinant will be:

$$\begin{vmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{vmatrix} = 0 \tag{15}$$

yielding the *characteristic equation*:

$$\left(2k - m\omega^2\right)^2 - k^2 = 0 \tag{16}$$

which has the solutions:

$$\omega_1 = \sqrt{\frac{k}{m}} \quad , \quad \omega_2 = \sqrt{\frac{3k}{m}} \tag{17}$$

Note that these values are the solutions to this particular case (masses are identical, springs are identical).To determine X_1 and X_2, we need to substitute into

$$\big[[K] - \omega^2[M]\big]\{x\} = \{0\}$$

with the values of ω_1 and ω_2 just obtained. Hence at $\omega = \omega_1 = \sqrt{\frac{k}{m}}$ we have:

$$\begin{bmatrix} 2k - m\omega_1^2 & -k \\ -k & 2k - m\omega_1^2 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = 0$$

This has infinite number of solutions, but they must satisfy a certain *ratio*, namely:

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_{\omega_1} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{18}$$

Similarly, at ω_2 we have:

$$\begin{bmatrix} 2k-m\omega_2^{\,2} & -k \\ -k & 2k-m\omega_2^{\,2} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = 0 \tag{19}$$

so

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_{\omega_2} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \tag{20}$$

The ratio of amplitudes $\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$ defines a certain *pattern* of motion called the **normal mode of vibration**. The vectors

$$\{X\}_{\omega_1} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_{\omega_1} \text{ and } \{X\}_{\omega_2} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_{\omega_2}$$

are called the **modal vectors** or **eigenvectors**. They define the **mode shapes** of the system. In this particular case, if the system vibrates in its first mode, the masses will move in phase with the same amplitudes, while in the second mode of vibration the masses move out of phase also with the same amplitudes.

The solution for the vibration of the system at the first mode is:

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = A_1 \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_{\omega_1} \cos(\omega_1 t + \phi_1) \tag{21}$$

and for the second mode:

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = A_2 \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_{\omega_2} \cos(\omega_2 t + \phi_2) \tag{22}$$

so the general solution is:

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = A_1 \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_{\omega_1} \cos(\omega_1 t + \phi_1) + A_2 \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_{\omega_2} \cos(\omega_2 t + \phi_2) \tag{23}$$

where A_1, A_2, ϕ_1 and ϕ_2 are 4 constants to be determined from the *initial conditions*.

Proof: It was noted that the solution of the equations

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0$$

$$m_2\ddot{x}_2 + (k_2 + k_3)x_2 - k_2 x_1 = 0$$

takes the form:

$$x_1 = X_1 \sin\omega t \quad , \quad x_2 = X_2 \sin\omega t$$

meaning the masses undergo harmonic motions of the **same frequency** with **no phase difference between them**. In order to justify this, let us re-write the equations of motions in a more general form:

$$\ddot{x}_1 + a_{11}x_1 + a_{12}x_2 = 0$$

$$\ddot{x}_2 + a_{21}x_1 + a_{22}x_2 = 0$$

Now assume a general solution in the form:

$$x_1 = X_1 \sin\omega_1 t \quad , \quad x_2 = X_2 \sin(\omega_2 t + \beta)$$

where ω_2 is taken to be different from ω_1.There is no loss of generality in assuming no phase for x_1 and only a phase difference β between the two motions. We wish to prove that $\omega_1 = \omega_2$ and $\beta = 0$.
Substituting into the equations of motion yields:

$$\left[a_{11} - \omega_1^2\right] X_1 \sin(\omega_1 t) + a_{12} X_2 \sin(\omega_2 t + \beta) = 0$$

$$a_{21} X_1 \sin(\omega_1 t) + \left[a_{22} - \omega_2^2\right] X_2 \sin(\omega_2 t + \beta) = 0$$

These relations must be valid for all t. Setting t=0 in the first equation gives:

$$a_{12} X_2 \sin(\beta) = 0$$

since a_{12} and X_2 cannot be zero, we must have $\beta = 0$. <u>Thus there can be no phase difference between the harmonic motions of the two parts</u>.
The first expression may then be written as:

$$\left[a_{11} - \omega_1^2\right] X_1 \sin(\omega_1 t) + a_{12} X_2 \sin(\omega_2 t) = 0$$

or

$$\frac{\sin(\omega_2 t)}{\sin(\omega_1 t)} = \frac{\left[\omega_1^2 - a_{11}\right] X_1}{a_{12} X_2} = \text{constant}$$

Since the left hand side must be constant for all values of t, we must have ω_2=ω_1 and consequently the <u>harmonic motions occur at the same frequency</u>.

Example

For various initial conditions, obtain the free response of the previous system having m = 1 and k = 1.

Solution

Recall the natural frequencies were

$$\omega_1 = \sqrt{\frac{k}{m}} = 1 \quad , \quad \omega_2 = \sqrt{\frac{3k}{m}} = \sqrt{3}$$

and the mode shapes were

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_{\omega_1} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad , \quad \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_{\omega_2} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

and the general solution is:

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = A_1 \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_{\omega_1} \sin(\omega_1 t + \phi_1) + A_2 \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_{\omega_2} \sin(\omega_2 t + \phi_2)$$

hence

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = A_1 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \sin(t + \phi_1) + A_2 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \sin\left(\sqrt{3}t + \phi_2\right)$$

Differentiating w.r.t. time we get:

$$\begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{Bmatrix} = A_1 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \cos(t + \phi_1) + \sqrt{3}A_2 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \cos\left(\sqrt{3}t + \phi_2\right)$$

For the initial conditions

$$x_1(0) = 5 \quad \dot{x}_1(0) = 0$$
$$x_2(0) = 0 \quad \dot{x}_2(0) = 0$$

we have:

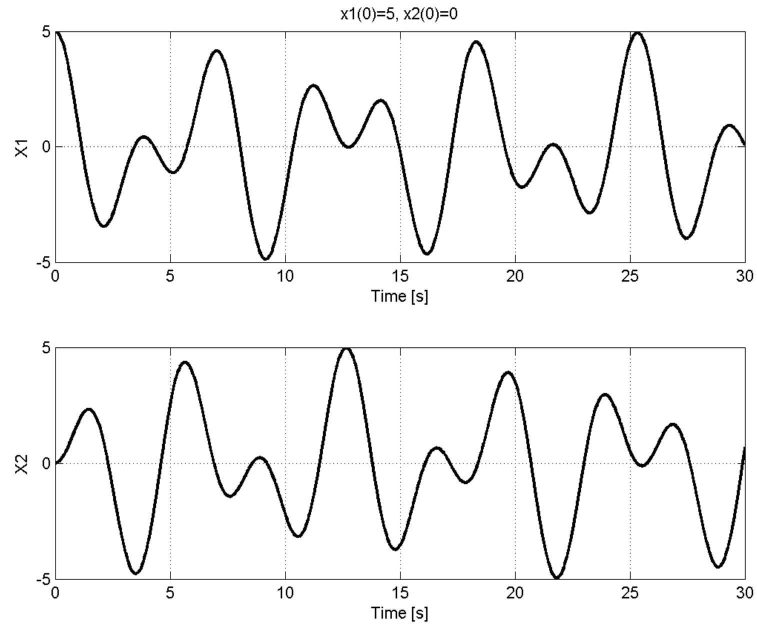
$$5 = A_1 \sin\phi_1 + A_2 \sin\phi_2$$
$$0 = A_1 \sin\phi_1 - A_2 \sin\phi_2$$
$$0 = A_1 \cos\phi_1 + \sqrt{3}A_2 \cos\phi_2$$
$$0 = A_1 \cos\phi_1 - \sqrt{3}A_2 \cos\phi_2$$

from which we get:

$$\phi_1 = \phi_2 = \pi/2$$
$$A_1 = A_2 = 5/2$$

hence the solution is:

$$x_1(t) = \frac{5}{2}\left[\cos t + \cos\sqrt{3}t\right] \quad , \quad x_2(t) = \frac{5}{2}\left[\cos t - \cos\sqrt{3}t\right]$$



Modes of vibration contribute equally to the solution.

For the initial conditions

$$x_1(0)=1 \quad \dot{x}_1(0)=0 \quad x_2(0)=1 \quad \dot{x}_2(0)=0$$

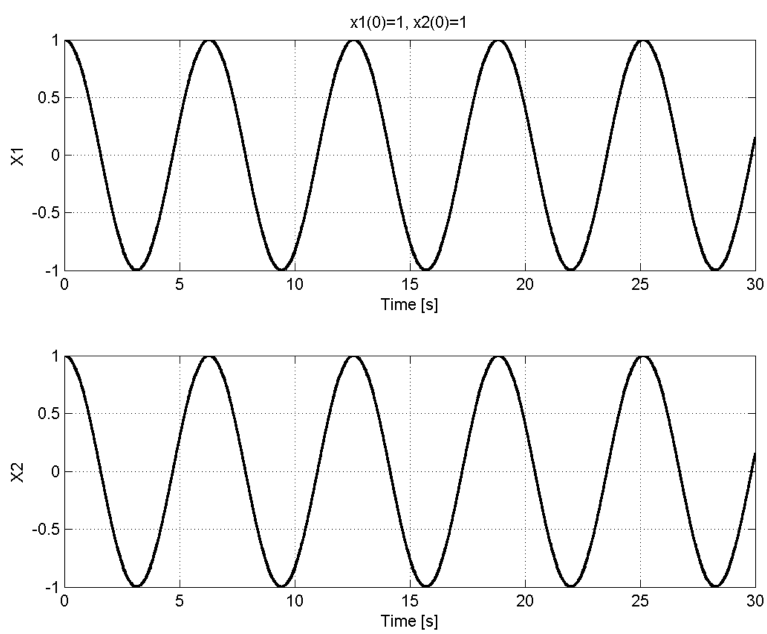
we get

$$\phi_1=\phi_2=\pi/2,\ A_1=1, A_2=0$$

hence the solution is:

$$x_1(t)=\cos t \quad , \quad x_2(t)=\cos t$$

i.e. the masses move in-phase with the same amplitude and frequency 1 rad /s (mode 1)



For the initial conditions

$$x_1(0)=1 \quad \dot{x}_1(0)=0 \quad x_2(0)=-1 \quad \dot{x}_2(0)=0$$

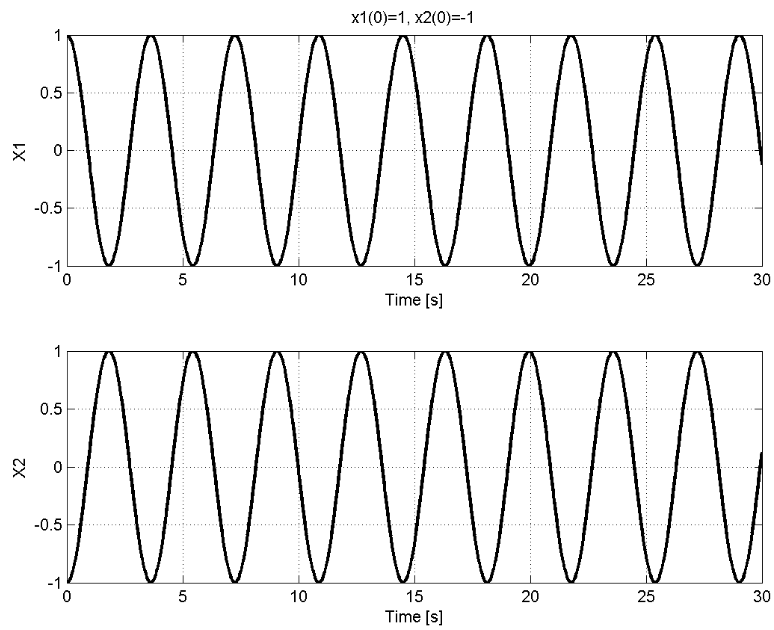
we get

$$\phi_1=\phi_2=\pi/2,\, A_1=0, A_2=1$$

hence the solution is:

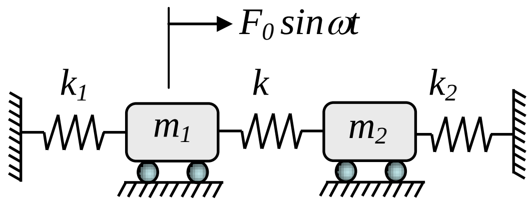
$$x_1(t)=\cos\sqrt{3}t \quad , \quad x_2(t)=-\cos\sqrt{3}t$$

i.e. the masses move out-of-phase with the same amplitude and frequency $\sqrt{3}$ rad /s (mode 2)



Forced vibration analysis

Consider the shown system



The equations of motion are:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k+k_1 & -k \\ -k & k+k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix} \sin \omega t \qquad (24)$$

or:

$$[M]\{\ddot{x}\} + [K]\{x\} = \{F\} \sin \omega t \qquad (25)$$

Seeking a steady-state solution in the form

$$\{x\} = \{X\} \sin \omega t \qquad (26)$$

yields:

$$\left[[K] - \omega^2 [M]\right]\{X\} \sin \omega t = \{F\} \sin \omega t \qquad (27)$$

hence:

$$\begin{bmatrix} k+k_1-\omega^2 m_1 & -k \\ -k & k+k_2-\omega^2 m_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix} \qquad (28)$$

which can be solved for the unknown amplitudes. For our special case where $m_1=m_2=m$ and $k_1=k_2=k$, we have:

$$\begin{bmatrix} 2k-\omega^2 m & -k \\ -k & 2k-\omega^2 m \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix} \qquad (29)$$

thus:

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{bmatrix} 2k-\omega^2 m & -k \\ -k & 2k-\omega^2 m \end{bmatrix}^{-1} \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix} \qquad (30)$$

This results in:

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \frac{1}{m^2\left(\omega^2 - \frac{k}{m}\right)\left(\omega^2 - \frac{3k}{m}\right)} \begin{Bmatrix} \left(2k-\omega^2 m\right) F_0 \\ kF_0 \end{Bmatrix} \qquad (31)$$

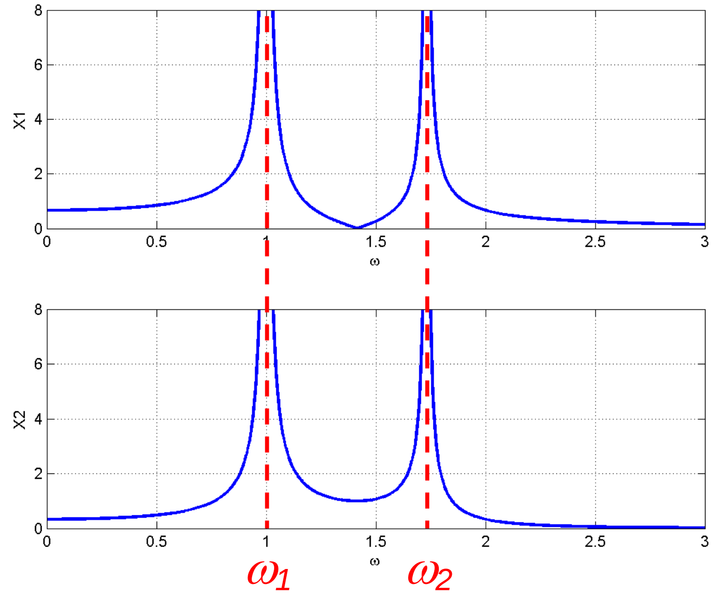
or:

$$X_1 = \frac{\left(2k-\omega^2 m\right) F_0}{m^2\left(\omega^2-\omega_1^2\right)\left(\omega^2-\omega_2^2\right)} \quad , \quad X_2 = \frac{kF_0}{m^2\left(\omega^2-\omega_1^2\right)\left(\omega^2-\omega_2^2\right)} \qquad (32)$$

where

$$\omega_1 = \sqrt{\frac{k}{m}} \quad , \quad \omega_2 = \sqrt{\frac{3k}{m}}$$

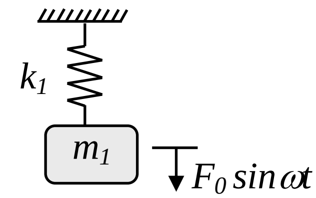
are the natural frequencies obtained earlier. Plotting the amplitudes of the masses reveals that resonance occurs when the frequency of excitation coincides with either of the two natural frequencies of the system.



X1
X2
ω
ω1
ω2

Dynamic vibration absorber

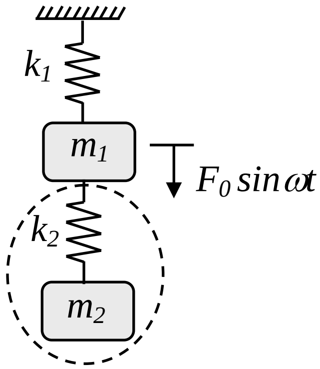
Consider the primary system shown, which is a model of a single DOF vibrating structure, acted upon by a harmonic force.



Suppose that the exciting frequency, ω, is constant and is equal to the natural frequency of the system, i.e.

$$\omega = \sqrt{\frac{k_1}{m_1}}$$

We wish to reduce the vibrations of m_1 at the exciting frequency ω. We can do this by adding a secondary system, consisting of a mass m_2 and spring k_2 as shown.



If we derive the equations of motion, we will get:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}\sin\omega t$$

Substituting into

$$\left[[K]-\omega^2[M]\right]\{X\}=\{F\} \quad \Rightarrow \quad \{X\}=\left[[K]-\omega^2[M]\right]^{-1}\{F\}$$

yields:

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{bmatrix} k_1+k_2-\omega^2 m_1 & -k_2 \\ -k_2 & k_2-\omega^2 m_2 \end{bmatrix}^{-1}\begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$$

or:

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \frac{1}{\left(k_1+k_2-\omega^2 m_1\right)\left(k_2-\omega^2 m_2\right)-k_2^2}\begin{bmatrix} k_2-\omega^2 m_2 & k_2 \\ k_2 & k_1+k_2-\omega^2 m_1 \end{bmatrix}\begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$$

hence:

$$X_1 = \frac{F_0\left(k_2-\omega^2 m_2\right)}{\left(k_1+k_2-\omega^2 m_1\right)\left(k_2-\omega^2 m_2\right)-k_2^2}$$

$$X_2 = \frac{F_0 k_2}{\left(k_1+k_2-\omega^2 m_1\right)\left(k_2-\omega^2 m_2\right)-k_2^2}$$

now define

$$\omega_{11} = \sqrt{\frac{k_1}{m_1}} \quad , \quad \omega_{22} = \sqrt{\frac{k_2}{m_2}}$$

For the primary system (without absorber), resonance occurs when

$$\omega = \sqrt{\frac{k_1}{m_1}} = \omega_{11}$$

For X_1 to be zero at this frequency, we must have

$$k_2 - \omega^2 m_2 = 0 \quad \Rightarrow \omega = \sqrt{\frac{k_2}{m_2}} = \omega_{22}$$

Therefore if k_2 and m_2 are chosen such that

$$\sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{k_2}{m_2}}$$

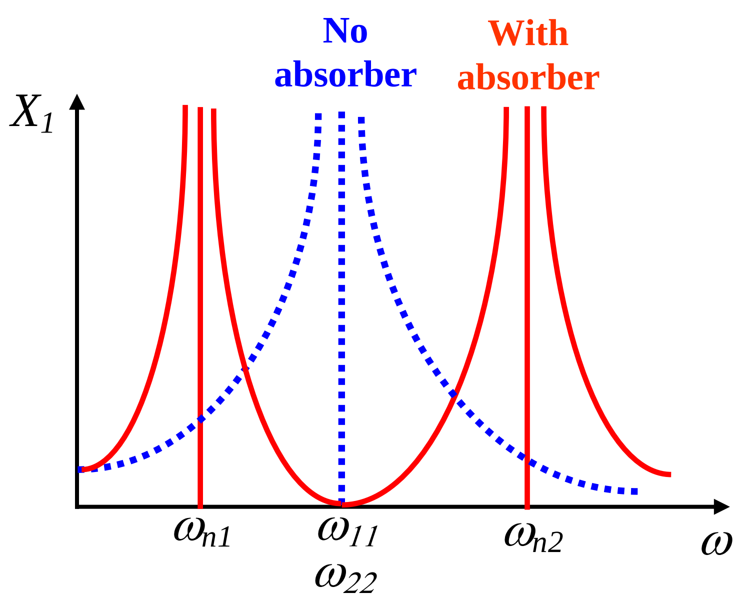
then X_1 will be zero at $\omega = \omega_{11}$. This is what we call a **tuned dynamic absorber**, in which

$$\omega_{11} = \omega_{22}$$

At this frequency, the displacement of X_2 will be:

$$X_2 = \frac{F_0 k_2}{\left(k_1 + k_2 - \frac{k_1}{m_1} m_1\right)\left(k_2 - \frac{k_2}{m_2} m_2\right) - k_2^2} = -\frac{F_0}{k_2}$$

Adding the secondary system (dynamic absorber) will result in zero vibrations of the primary mass at $\omega_{11} = \omega_{22}$. However, two resonant frequencies ω_{n1} and ω_{n2} are introduced at which the amplitude of X_1 becomes significantly large. Thus the dynamic absorber can only be useful when the disturbing frequency is constant.



How to design the vibration absorber?

1. Based on space limitation. Choose k_2 and m_2 such that

$$\sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{k_2}{m_2}}$$

noting that the amplitude of m_2 will be

$$X_2 = -\frac{F_0}{k_2}$$

2. Based on how far apart should the natural frequencies be. The two new natural frequencies of the system can be obtained by setting the determinant equal to zero:

$$\begin{vmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{vmatrix} = 0$$

which gives:

$$\left(k_1 + k_2 - \omega^2 m_1\right)\left(k_2 - \omega^2 m_2\right) - k_2^2 = 0$$

This gives:

$$\omega^4 - \omega^2\left(\frac{k_1}{m_1} + \frac{k_2}{m_1} + \frac{k_2}{m_2}\right) + \frac{k_1 k_2}{m_1 m_2} = 0$$

which can be put in the form:

$$\omega^4 - \omega^2\left(\omega_{11}^2 + \mu\omega_{22}^2 + \omega_{22}^2\right) + \omega_{11}^2\omega_{22}^2 = 0$$

where $\mu = \frac{m_2}{m_1} = \text{mass ratio}$

The roots of this equation ω_{n1} and ω_{n2} satisfy the relations:

$$\omega_{n1}^2 \cdot \omega_{n2}^2 = \omega_{11}^2 \cdot \omega_{11}^2$$

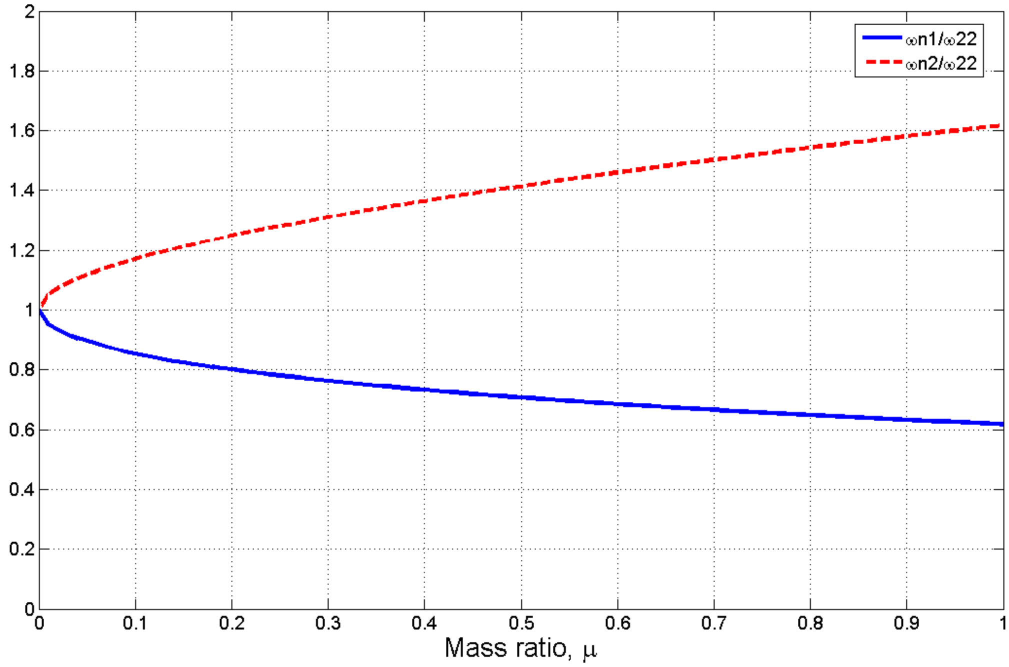
$$\omega_{n1}^2 + \omega_{n2}^2 = \omega_{11}^2 + \omega_{22}^2\left(1 + \mu\right)$$

But for a tuned absorber we have $\omega_{11} = \omega_{22}$ hence

$$\frac{\omega_{n1}^2}{\omega_{22}^2} \cdot \frac{\omega_{n2}^2}{\omega_{22}^2} = 1$$

$$\frac{\omega_{n1}^2}{\omega_{22}^2} + \frac{\omega_{n2}^2}{\omega_{22}^2} = 2 + \mu$$

As you increase the mass ratio (μ), the natural frequencies ω_{n1} and ω_{n2} will grow further apart. Note that ω_{n1} is always closer to ω_{11} than ω_{n2}.



ωn1/ω22
ωn2/ω22
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Mass ratio, μ